LOWER BOUNDS AROUND THE RANK $2G-1$ FINITE-ORBIT CONSTRUCTION

ABSTRACT. Let $\Sigma_{g,n}$ be an oriented genus g surface with n punctures. Landesman and Litt proved that every mapping-class-group-finite complex representation of $\pi_1(\Sigma_{g,n})$ of rank $< \sqrt{g+1}$ has finite image. A recent comment on Litt's Problem 10 records a smaller non-semisimple construction of rank $2g-1$, using a non-trivial finite character and the full twisted cohomology $H^1(\Sigma_g, \mathbb{C}_\chi)$. We take that construction as an input and do not reproduce it here.

The purpose of this note is to record lower bounds around that construction. First, using Looijenga's big monodromy theorem for abelian Prym representations, we prove that the rank $2g-1$ bound is sharp for closed surfaces, $g \geq 3$, among MCG-finite representations whose semisimplification is a sum of finite characters. In the scalar-unipotent subcase the sharp bound is $2g+1$. Second, for arbitrary finite-image semisimplification, we sharpen the non-semisimple step in Landesman–Litt: if ρ^{ss} has finite image and $\operatorname{rk}\rho < 2\sqrt{g}$, then ρ has finite image. Equivalently,

$$\lceil 2\sqrt{g} \rceil \leq m_{\mathrm{fss}}(g,n) \leq 2g-1.$$

Finally, combining the rank-three instance of the construction with the known rank-two classification gives $m(2,n) = 3$ for every $n \geq 0$. The semisimple minimum remains open.

1. INTRODUCTION AND STATEMENTS

Put

$$\Pi_{g,n} := \pi_1(\Sigma_{g,n}), \qquad \Pi_g := \Pi_{g,0}.$$

The mapping class group $\mathrm{Mod}_{g,n}$ acts on conjugacy classes of representations

$$\rho : \Pi_{g,n} \longrightarrow \mathrm{GL}_r(\mathbb{C})$$

through its outer action on $\Pi_{g,n}$. Following Landesman–Litt [3], call ρ *MCG-finite* if its conjugacy class has finite $\mathrm{Mod}_{g,n}$-orbit. Define

$$m(g,n) := \min\{r : \exists \text{ an MCG-finite } \rho : \Pi_{g,n} \to \mathrm{GL}_r(\mathbb{C}) \text{ with infinite image}\},$$

and define $m^{\mathrm{ss}}(g,n)$ similarly, with ρ required to be semisimple. We also write

$$m_{\mathrm{fss}}(g,n)$$

for the same minimum under the condition that ρ^{ss} have finite image.

Landesman and Litt prove the general lower bound

$$r < \sqrt{g+1} \quad \Longrightarrow \quad \rho(\Pi_{g,n}) \text{ is finite} \tag{1}$$

for every MCG-finite representation $\rho : \Pi_{g,n} \to \mathrm{GL}_r(\mathbb{C})$ [3, Theorem 1.2.1]. They also record a $(2g+1)$-dimensional non-semisimple construction with infinite image [3, Example 10.1.6(3), Remark 10.2.2]. The following smaller upper bound is the twisted affine construction described in the comment thread to Litt's Problem 10 [7]. We use only its statement.

Proposition 1.1 (Known rank $2g-1$ construction)**.** *For every $g \geq 2$ and $n \geq 0$, there exists an MCG-finite representation*

$$\Pi_{g,n} \longrightarrow \mathrm{GL}_{2g-1}(\mathbb{C})$$

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with infinite image and finite-image semisimplification. Hence

$$m(g,n) \le m_{\mathrm{fss}}(g,n) \le 2g-1.$$

For closed surfaces the construction may be chosen with semisimplification

$$\chi \oplus \mathbf{1}^{\oplus(2g-2)}$$

for a non-trivial finite character $\chi : \Pi_g \to \mathbb{C}^\times$.

The first lower-bound result is an exact closed-surface statement for finite abelian semisimplification.

Theorem 1.2 (Closed surfaces with finite abelian semisimplification)**.** *Let $g \ge 3$, and let*

$$\rho : \Pi_g \longrightarrow \mathrm{GL}_r(\mathbb{C})$$

be MCG-finite. Suppose that every Jordan–Holder constituent of ρ is a finite character. If ρ has infinite image, then

$$r \ge 2g-1.$$

The bound is sharp by Proposition 1.1. Hence the minimum rank of an infinite-image MCG-finite representation of Π_g with finite abelian semisimplification is exactly $2g-1$.

The scalar-unipotent subcase has the older $(2g+1)$-bound, and this is also sharp.

Corollary 1.3 (Scalar-unipotent case)**.** *Let $g \ge 3$, and let $\rho : \Pi_g \to \mathrm{GL}_r(\mathbb{C})$ be MCG-finite. Suppose that all Jordan–Holder constituents of ρ are equal to a single finite character. If ρ has infinite image, then*

$$r \ge 2g+1.$$

This bound is sharp, up to twisting by a finite character, by Morita's $(2g+1)$-dimensional unipotent construction.

The second lower-bound result applies to arbitrary finite-image semisimplification. It improves the bound obtained by keeping only the isotypic-dimension estimate in the Landesman–Litt proof. The point is to apply their cohomological theorem directly to the multiplicity-space image of the extension class.

Theorem 1.4 (Cohomological extension obstruction)**.** *Let $g \ge 2$, $n \ge 0$, and let*

$$0 \longrightarrow \rho_1 \longrightarrow \rho \longrightarrow \rho_2 \longrightarrow 0$$

be an MCG-finite extension of representations of $\Pi_{g,n}$. Assume that ρ_1 is characteristic in ρ, that ρ_1 and ρ_2 are semisimple, and that ρ_1, ρ_2 have finite image. Write

$$\rho_2^\vee \otimes \rho_1 \simeq \bigoplus_i \sigma_i \otimes M_i,$$

where the σ_i are pairwise non-isomorphic irreducible finite-image representations and M_i is a multiplicity space of dimension n_i. If the extension is non-split, then for some i,

$$n_i \ge 2g - 2\dim \sigma_i. \tag{2}$$

Consequently, if $a = \dim \rho_1$ and $b = \dim \rho_2$, then a non-split extension forces

$$ab \ge g. \tag{3}$$

Corollary 1.5 (Finite-image semisimplification)**.** *Let $g \geq 2$, $n \geq 0$, and let*

$$\rho : \Pi_{g,n} \longrightarrow \mathrm{GL}_r(\mathbb{C})$$

be MCG-finite. If ρ^{ss} has finite image and

$$r < 2\sqrt{g},$$

then ρ has finite image. Equivalently,

$$m_{\mathrm{fss}}(g,n) \geq \lceil 2\sqrt{g} \rceil.$$

Together with Proposition 1.1, this gives

$$\boxed{\lceil 2\sqrt{g} \rceil \leq m_{\mathrm{fss}}(g,n) \leq 2g-1} \qquad (g \geq 2).$$

In genus two the upper bound in Proposition 1.1 has rank three, and known rank-two results rule out anything smaller.

Corollary 1.6. *For every $n \geq 0$,*

$$m(2,n) = 3.$$

The semisimple problem is not settled here. Proposition 1.1 is non-semisimple, and Corollary 1.5 says only that an example below $2\sqrt{g}$ would have to have infinite-image semisimplification.

2. Two standard inputs

We isolate the two external inputs used in the lower bounds.

2.1. Landesman–Litt's cohomological lower bound.

We use the following form of [3, Theorem 1.7.1].

Theorem 2.1 (Landesman–Litt)**.** *Let $g \geq 2$, $n \geq 0$, and let*

$$\sigma : \Pi_{g,n} \longrightarrow \mathrm{GL}_d(\mathbb{C})$$

be an irreducible finite-image representation. Let Γ be a finite-index subgroup of the mapping class group, after adding a base point if necessary, which fixes σ as a representation. Then every nonzero Γ-invariant subspace of $H^1(\Pi_{g,n}, \sigma)$ has dimension at least

$$2g - 2d.$$

When $2g - 2d \leq 0$, the assertion is vacuous.

Proof. The subgroup fixing σ as a homomorphism defines a unitary local system $\mathcal{V}$ on the universal punctured curve over the corresponding finite etale cover of $\mathcal{M}_{g,n}$. The fiber of $R^1\pi_*^\circ\mathcal{V}$ is $H^1(\Pi_{g,n}, \sigma)$. Hence an invariant subspace is the fiber of a sub-local system of $R^1\pi_*^\circ\mathcal{V}$. Landesman–Litt's Theorem 1.7.1 says that every nonzero such sub-local system has rank at least $2g - 2\,\mathrm{rk}\,\mathcal{V} = 2g - 2d$. □

Remark 2.2. This formulation is deliberately a statement about invariant subspaces of the cohomology representation. It is not a statement about arbitrary subquotients of the homology of punctured finite covers; Landesman–Litt explicitly warn that the relevant homology representations need not be semisimple in the punctured case [3, Remark 7.2.2].

2.2. **Prym monodromy for finite characters.** The sharp abelian lower bound uses the following standard consequence of Looijenga's theorem on Prym representations.

Proposition 2.3 (Prym monodromy)**.** *Let $g \geq 3$. For a finite character $\lambda : \Pi_g \to \mathbb{C}^\times$, put*

$$V_\lambda := H^1(\Pi_g, \mathbb{C}_\lambda).$$

Then:

(i) *If $\lambda = 1$, then $\dim V_\lambda = 2g$, and the virtual mapping class group action on V_λ is Zariski dense in $\mathrm{Sp}(V_\lambda)$.*

(ii) *If $\lambda \neq 1$, then $\dim V_\lambda = 2g-2$. The virtual stabilizer of λ has Zariski-dense image in the natural classical group acting on V_λ: namely $\mathrm{Sp}(V_\lambda)$ when λ is self-dual, and a group containing $\mathrm{SL}(V_\lambda)$ as its derived complex group when λ is not self-dual.*

Proof. Part (i) is the classical symplectic representation of Mod_g. Part (ii) is the one-character case of Looijenga's computation of the Zariski closures of Prym representations for finite abelian covers of closed surfaces [5, Corollary 2.6]. In the terminology of Landesman–Litt–Sawin, the abelian case of their big-monodromy theorem is exactly Looijenga's theorem, and for abelian covering group the genus hypothesis reduces to $g \geq 3$ [4, Remark 1.4]. □

3. The general finite-semisimplification obstruction

We prove Theorem 1.4 and Corollary 1.5.

Proof of Theorem 1.4. Let

$$\tau := \rho_2^\vee \otimes \rho_1.$$

The extension class of ρ is an element

$$e \in H^1(\Pi_{g,n}, \tau).$$

Using

$$\tau \simeq \bigoplus_i \sigma_i \otimes M_i,$$

we have

$$H^1(\Pi_{g,n}, \tau) \simeq \bigoplus_i H^1(\Pi_{g,n}, \sigma_i) \otimes M_i.$$

Write e_i for the component of e in the i-th summand. Since the extension is non-split, $e_i \neq 0$ for some i. Fix such an i, and regard

$$e_i \in H^1(\Pi_{g,n}, \sigma_i) \otimes M_i$$

as a linear map

$$\phi_i : M_i^\vee \longrightarrow H^1(\Pi_{g,n}, \sigma_i).$$

Let

$$S_i := \mathrm{im}\, \phi_i.$$

Then $S_i \neq 0$ and $\dim S_i \leq n_i$.

We claim that, after passing to a finite-index subgroup of the mapping class group, S_i is invariant. To make the action honest, take the preimage of the stabilizer in the mapping class group with one additional marked base point. Then pass to a finite-index subgroup stabilizing the conjugacy class of ρ, the isomorphism classes of ρ_1, ρ_2, and σ_i, and finally fixing the finite representation σ_i itself. Because ρ_1 is characteristic, any isomorphism between ρ and a mapping-class pullback of ρ preserves ρ_1 and induces an isomorphism on ρ_2. Thus the extension class is carried to itself up to the natural action of $\mathrm{Aut}(\rho_1) \times \mathrm{Aut}(\rho_2)$ on τ. On the

σ_i-isotypic summand, this action is through $\mathrm{GL}(M_i)$ on the multiplicity factor. Consequently ϕ_i changes by precomposition with an automorphism of $M_i^\vee$, and its image S_i is preserved.

By Theorem 2.1,

$$\dim S_i \geq 2g - 2\dim \sigma_i.$$

Since $\dim S_i \leq n_i$, this proves (2).

It remains to deduce $ab \geq g$. Put $d_i = \dim \sigma_i$. Since $\sigma_i \otimes M_i$ is a summand of the ab-dimensional representation τ,

$$n_i d_i \leq ab. \tag{4}$$

If $d_i \geq g$, then $ab \geq d_i \geq g$. If $d_i < g$, then (2) and (4) give

$$ab \geq n_i d_i \geq d_i(2g - 2d_i) = 2d_i(g - d_i).$$

For an integer $1 \leq d_i \leq g-1$, the last quantity is at least $2g-2$, hence at least g because $g \geq 2$. Thus $ab \geq g$ in all cases. □

Proof of Corollary 1.5. We argue by induction on the length of the socle filtration. If ρ is semisimple, then $\rho = \rho^{\mathrm{ss}}$ has finite image by assumption.

Let $A = \mathrm{Soc}(\rho)$. It is characteristic and semisimple; since ρ^{ss} has finite image, A has finite image. The quotient $Q = \rho/A$ is MCG-finite and has finite-image semisimplification. By induction, Q has finite image. Over $\mathbb{C}$, finite-image representations are semisimple, so Q is semisimple of finite image.

If the extension

$$0 \to A \to \rho \to Q \to 0$$

were non-split, Theorem 1.4 would give

$$\dim A \cdot \dim Q \geq g.$$

But

$$\dim A \cdot \dim Q \leq \frac{(\dim A + \dim Q)^2}{4} = \frac{r^2}{4} < g,$$

a contradiction. Hence the extension splits. Therefore $\rho \simeq A \oplus Q$ has finite image. □

Remark 3.1. In [3, Lemma 8.6.1], the corresponding argument passes through a map from the homology of a finite cover trivializing $\rho_1 \oplus \rho_2$ to $\rho_2^\vee \otimes \rho_1$. If one keeps only the dimension of the whole isotypic target, one obtains the weaker estimate

$$n_i \dim \sigma_i \geq 2g - 2\dim \sigma_i.$$

The improvement above is the replacement of the isotypic target dimension $n_i \dim \sigma_i$ by the multiplicity-space dimension n_i, using Theorem 2.1 directly.

4. Closed surfaces with finite abelian semisimplification

This section proves Theorem 1.2 and Corollary 1.3. The key representation-theoretic point is that the standard representation of SL_N or Sp_N cannot be hidden inside a tensor product of two smaller representations.

Lemma 4.1 (Tensor recognition)**.** *Let G be either $\mathrm{SL}(V)$ or $\mathrm{Sp}(V)$, with $N := \dim V \geq 2$, and let $\Gamma \subset G(\mathbb{C})$ be Zariski dense. Let $A_1, \dots, A_s$ and $B_1, \dots, B_t$ be finite-dimensional complex vector spaces, and set*

$$L := \prod_i \mathrm{GL}(A_i) \times \prod_j \mathrm{GL}(B_j).$$

For a set E of pairs (i,j), put

$$W := \bigoplus_{(i,j)\in E} \mathrm{Hom}(B_j, A_i).$$

Suppose $0 \neq e \in V \otimes W$ has the following property: for every $\gamma \in \Gamma$, there is $\ell_\gamma \in L$ such that

$$(\gamma, \ell_\gamma)e = e.$$

Then some pair (i,j) occurring in the support of e satisfies

$$\dim A_i + \dim B_j \geq N+1.$$

Proof. Let $S \subset G \times L$ be the stabilizer of e. The projection $S \to G$ contains the Zariski-dense subgroup Γ, hence is dominant. Passing to identity components, the projection $S^\circ \to G$ is surjective. Its unipotent radical maps trivially, because the image would be a connected normal unipotent subgroup of the semisimple group G. Hence a Levi factor $M \subset S^\circ$ still maps onto G. Since G is simple, some simple factor of M maps isogenously onto G.

Restrict to such a factor. The vector spaces A_i and B_j become algebraic representations of a group isogenous to G, and e is an invariant tensor. Equivalently, $V^\vee$ occurs as a subrepresentation of

$$W = \bigoplus_{(i,j)\in E} A_i \otimes B_j^\vee.$$

By semisimplicity, $V^\vee$ occurs in one summand $A_i \otimes B_j^\vee$.

For a group isogenous to SL_N or Sp_N in characteristic zero, every non-trivial representation has dimension at least N. Thus, if both $\dim A_i < N$ and $\dim B_j < N$, both are trivial representations of the simple factor, so their tensor product cannot contain $V^\vee$. Hence at least one of A_i, B_j has dimension at least N, and the other is nonzero. Therefore $\dim A_i + \dim B_j \geq N+1$. □

Lemma 4.2 (One nonsplit abelian layer)**.** *Let $g \geq 3$, and let*

$$0 \longrightarrow A \longrightarrow E \longrightarrow B \longrightarrow 0$$

be a non-split MCG-finite extension of Π_g-representations. Assume that A is characteristic in E, and that A and B are semisimple with finite abelian image. Then

$$\dim E \geq 2g-1.$$

If all characters appearing in $A \oplus B$ are equal, then

$$\dim E \geq 2g+1.$$

Proof. Decompose

$$A = \bigoplus_\alpha \alpha \otimes A_\alpha, \qquad B = \bigoplus_\beta \beta \otimes B_\beta,$$

where α, β range over finite characters and A_α, B_β are multiplicity spaces. The extension class lies in

$$\mathrm{Ext}^1(B, A) \simeq \bigoplus_{\alpha,\beta} H^1(\Pi_g, \alpha\beta^{-1}) \otimes \mathrm{Hom}(B_\beta, A_\alpha). \tag{5}$$

Pass to a finite-index subgroup $\Gamma \subset \mathrm{Mod}_g$ stabilizing all characters appearing in $A \oplus B$ and stabilizing the conjugacy class of E. Intersect further with finite-index subgroups supplied by Proposition 2.3 for every ratio $\alpha\beta^{-1}$ that occurs; Zariski density is unchanged by this finite-index replacement. Since A is characteristic, conjugacies preserving the extension preserve A

and act on the multiplicity spaces. Thus the Γ-orbit of the extension class is contained in the orbit of

$$L := \prod_{\alpha} \mathrm{GL}(A_\alpha) \times \prod_{\beta} \mathrm{GL}(B_\beta).$$

Because the extension is non-split, some component of (5) is nonzero. Group all nonzero components with a fixed ratio

$$\lambda = \alpha\beta^{-1}.$$

This gives a nonzero tensor

$$e_\lambda \in V_\lambda \otimes \bigoplus_{\alpha\beta^{-1}=\lambda} \mathrm{Hom}(B_\beta, A_\alpha), \qquad V_\lambda := H^1(\Pi_g, \mathbb{C}_\lambda),$$

whose Γ-orbit is contained in the corresponding L-orbit. Proposition 2.3 gives a Zariski-dense action of $\mathrm{SL}(V_\lambda)$ or $\mathrm{Sp}(V_\lambda)$, after replacing by the derived classical subgroup in the non-self-dual case. Lemma 4.1 therefore gives a pair (α, β) in the support of e_λ such that

$$\dim A_\alpha + \dim B_\beta \geq \dim V_\lambda + 1.$$

If $\lambda \neq 1$, then $\dim V_\lambda = 2g-2$, so $\dim E \geq 2g-1$. If $\lambda = 1$, then $\dim V_\lambda = 2g$, so $\dim E \geq 2g+1$. The second assertion follows because, when all characters in $A \oplus B$ are equal, every ratio $\alpha\beta^{-1}$ is 1. □

Proof of Theorem 1.2. We argue by induction on r. The semisimple case has finite image, so there is nothing to prove. Let $A = \mathrm{Soc}(\rho)$. This is characteristic, semisimple, and has finite abelian image. The quotient $Q = \rho/A$ is MCG-finite and again has finite abelian semisimplification.

If Q has infinite image, the induction hypothesis gives $\dim Q \geq 2g-1$, hence $r \geq 2g-1$. If Q has finite image, then Q is semisimple over $\mathbb{C}$. Since ρ has infinite image, the extension

$$0 \longrightarrow A \longrightarrow \rho \longrightarrow Q \longrightarrow 0$$

is non-split. Lemma 4.2 gives $r \geq 2g-1$. Sharpness is Proposition 1.1. □

Proof of Corollary 1.3. The proof is identical to the proof of Theorem 1.2, but uses the second assertion of Lemma 4.2. Indeed, if all Jordan–Holder constituents of ρ are equal to one finite character, then the same is true for the socle and for every quotient appearing in the induction. Hence the first nonsplit layer over a finite-image quotient has rank at least $2g+1$. Morita's construction gives the sharp $(2g+1)$-dimensional example [3, Example 10.1.6(3)]. □

5. Genus two and remaining gaps

Proof of Corollary 1.6. Proposition 1.1 gives $m(2, n) \leq 3$. Rank one is impossible in positive genus by [3, Proposition 2.3.1]. Rank two is also impossible for infinite image in genus at least two. In the semisimple case, the determinant is finite by [3, Lemma 2.3.2]; after a finite character twist, this reduces to the $\mathrm{SL}_2(\mathbb{C})$ case, where Biswas–Gupta–Mj–Whang prove finite image for genus at least two [1, Theorem A]. In the reducible non-semisimple case, Cousin–Heu's classification of reducible rank-two finite orbits in positive genus shows that the only infinite-image reducible finite-orbit examples occur when $g = 1$ and $n > 0$ [2, Theorem B]. Hence $m(2, n) = 3$. □

The results above give the following closed-surface picture for $g \geq 3$:

class of representations	sharp minimum rank for infinite image
finite abelian semisimplification	$2g-1$
scalar/unipotent semisimplification	$2g+1$.

For arbitrary finite-image semisimplification, Corollary 1.5 gives

$$\lceil 2\sqrt{g} \rceil \leq m_{\mathrm{fss}}(g,n) \leq 2g-1.$$

The remaining finite-semisimplification gap is therefore genuinely non-abelian: a smaller example would need a nonsplit layer whose coefficient $\rho_2^\vee \otimes \rho_1$ contains higher-dimensional finite local systems, and it would have to come close to the lower bound in Theorem 2.1.

The full problem is still bottlenecked by semisimple representations. If ρ^{ss} has infinite image, the finite-semisimplification arguments do not apply, and the general lower bound remains (1). Landesman–Litt record semisimple infinite-image examples of rank exponential in g, coming from TQFT and Kodaira–Parshin-type constructions [3, Example 10.2.4]. No polynomial-rank semisimple examples are known from their discussion.

For punctured surfaces, Proposition 1.1 still gives $m(g,n) \leq 2g-1$, and Corollary 1.6 gives $m(2,n) = 3$. The closed-surface sharp lower bound in Theorem 1.2 does not automatically extend to arbitrary punctured representations, because punctured cohomology has mixed boundary contributions. The cohomological lower bound in Theorem 2.1 remains valid and is what gives Corollary 1.5 for all n.

Question 5.1. *Is $m_{\mathrm{fss}}(g,n) = 2g-1$ for all $g \geq 2$?*

Question 5.2. *Can the semisimple lower bound be improved beyond $r < \sqrt{g+1}$, for example to a polynomial or linear range in g?*

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