A MICROLOCAL SIEVE FOR EXCEPTIONAL TANNAKA GROUPS OF SUBVARIETIES OF ABELIAN VARIETIES

ABSTRACT. Let A be a complex abelian variety and let $X \subset A$ be an integral subvariety. The convolution tensor category generated by the perverse intersection complex

$$\delta_X = \mathrm{IC}_X[\dim X]$$

modulo negligible objects has a Tannaka group. We develop a microlocal sieve for the possibility that the connected derived Tannaka group attached to δ_X has exceptional type

$$G_2, \qquad F_4, \qquad E_7, \qquad E_8.$$

The sieve is strongest for singular candidates, where the usual minuscule obstruction for smooth subvarieties no longer applies directly.

The main results are the following. If the clean characteristic cycle of δ_X is irreducible, then the Tannaka representation is minuscule; hence G_2, F_4, E_8 are impossible and E_7 is forced into its 56-dimensional representation. If all nonzero weights of the representation form a single Weyl orbit, then the representation is minuscule or quasi-minuscule, leaving for the four target types only

$$G_2 \curvearrowright 7, \quad F_4 \curvearrowright 26, \quad E_7 \curvearrowright 56, 133, \quad E_8 \curvearrowright 248.$$

For every irreducible representation V of an exceptional group H we prove the congruence

$$\dim V - z(V) \equiv 0 \pmod{m_H}, \qquad (m_{G_2}, m_{F_4}, m_{E_7}, m_{E_8}) = (6, 24, 2, 240),$$

where $z(V)$ is the zero-weight multiplicity. Microlocally this is a polar-degree divisibility condition on the nonzero-weight part of the clean characteristic cycle.

For divisors with isolated ordinary double points, Krämer's formula for characteristic cycles converts the representation-theoretic sieve into explicit numerical restrictions. In odd ambient dimension an ordinary double point contributes one point-conormal component. Without further hypotheses this gives degree and parity constraints; under a natural zero-separation hypothesis it gives the exact node counts $1, 2, 7, 8$ for the one-nonzero-orbit cases G_2, F_4, E_7, E_8. Under Krämer's torsion-separation hypothesis the divisor is essentially multiplicity-free, which rules out F_4 and E_8 outright and rules out E_7 in odd ambient dimension.

Finally, low-degree invariant tensors give geometric tests on addition maps $X^n \to A$. The cubic invariants for G_2, F_4 and adjoint E_7, E_8 force punctual summands in degree three when the full Tannaka central character is trivial in degree three, and always in degree six after squaring. The quartic invariant of the 56 of E_7 gives a degree-four test.

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1. Introduction

Let A be a complex abelian variety. The group law $a : A \times A \to A$ defines the convolution product

$$K * L = Ra_*(K \boxtimes L)$$

on constructible complexes. Krämer and Weissauer showed that, after passing to the quotient by negligible objects, convolution endows suitable subcategories of perverse sheaves on A with a neutral Tannakian structure [10]. Thus an integral subvariety $X \subset A$ gives a tensor generator

$$\delta_X = \mathrm{IC}_X[\dim X]$$

and hence a reductive Tannaka group. We write

$$G_{X,\omega}$$

for the full group attached to a choice of fiber functor ω, and

$$G^*_{X,\omega} = (G^\circ_{X,\omega})_{\mathrm{der}}$$

for its connected derived group. The fiber functor gives a finite-dimensional representation

$$V_X = \omega(\delta_X).$$

When X is non-negligible and nondivisible, $G^\circ_{X,\omega}$ acts irreducibly on V_X in the standard smooth setup; in the singular situation we state irreducibility wherever it is used.

A fundamental example is the Fano surface $F(Y)$ of lines on a smooth cubic threefold $Y \subset \mathbb{P}^4$. After choosing a base point, the Albanese map embeds $F(Y)$ into its five-dimensional Albanese variety, and the corresponding Tannaka group has connected derived group E_6 acting through a 27-dimensional minuscule representation [6]. Krämer–Lehn–Maculan recently proved that, under smoothness, positivity and high-codimension hypotheses, this cubic-threefold construction is the only exceptional case [9].

The natural question is whether the other exceptional types

$$G_2, F_4, E_7, E_8$$

can occur as $G^*_{X,\omega}$ for some subvariety $X \subset A$. In the smooth case, the answer is highly constrained because the representation V_X is minuscule. Among exceptional simple groups the only nontrivial minuscule representations are the two 27-dimensional representations of E_6 and the 56 of E_7. Thus G_2, F_4, E_8 cannot arise from a smooth subvariety in the minuscule range, and E_7 is forced into the 56.

The purpose of this paper is to push the obstruction into the singular regime. The correct language is microlocal. Let

$$\mathrm{cc}(K) = \sum_i n_i \Lambda_i$$

be the clean characteristic cycle of a perverse sheaf or regular holonomic $\mathscr{D}$-module K on A: degenerate components are discarded, and each remaining irreducible conic Lagrangian component has a generically finite Gauss map. The clean degree is

$$\deg \mathrm{cc}(K) = \sum_i n_i \deg(\Lambda_i).$$

The Franecki–Kapranov index formula and generic vanishing give

$$\deg \mathrm{cc}(K) = \chi(A, K) = \dim \omega(K)$$

for the clean representative of a non-negligible perverse sheaf. Krämer's microlocalization theorem relates the clean cycle to the Weyl-group orbits of weights of the Tannaka representation [7, 8]. We exploit this relation in four ways: irreducibility, orbit arithmetic, nodal formulas, and invariant tensors.

Main statements. The first obstruction is the clean-cycle version of the minuscule phenomenon.

Theorem 1.1 (Irreducible clean cycle obstruction)**.** *Let $X \subset A$ be integral and non-negligible, and assume that V_X is irreducible for $G^*_{X,\omega}$. If $\mathrm{cc}(\delta_X)$ is irreducible and $G^*_{X,\omega}$ is simple exceptional, then V_X is minuscule. Consequently G_2, F_4, E_8 do not occur in this situation, and E_7 can occur only through its 56-dimensional representation.*

Thus any example of type G_2, F_4 or E_8 must have genuinely reducible clean characteristic cycle. The next obstruction concerns the first reducible case: one nonzero Weyl orbit, with possible zero weights.

Theorem 1.2 (One-nonzero-orbit sieve)**.** *Let H be simple exceptional and let V be a nontrivial irreducible representation of H. If all nonzero weights of V form one Weyl-group orbit, then V is minuscule or quasi-minuscule. For the four target types the only possibilities are*

H	V	*weight-degree pattern*
G_2	7	$6 + 1$
F_4	26	$24 + 2$
E_7	56	56
E_7	133	$126 + 7$
E_8	248	$240 + 8$.

The number after the plus sign is the zero-weight multiplicity.

Even when there are many nonzero orbits, there is a useful divisibility test.

Theorem 1.3 (Zero-weight congruence)**.** *Let V be an irreducible representation of a simple exceptional group H, and let $z(V) = \dim V_0$ be the zero-weight multiplicity. Then*

$$\dim V - z(V) \equiv 0 \pmod{m_H},$$

where

H	G_2	F_4	E_6	E_7	E_8
m_H	6	24	9	2	240.

For $K = \delta_X$, the same congruence holds for the clean degree of the nonzero-weight microlocal summand.

The singular class where this becomes most concrete is the class of nodal divisors. If $D \subset A$ is a divisor with isolated ordinary double points and $g = \dim A$ is odd, Krämer's formula says that each node contributes one point-conormal component to $\mathrm{CC}(\delta_D)$; if g is even, these point-conormal components do not occur. This yields explicit restrictions.

Theorem 1.4 (Nodal sieve)**.** *Let $D \subset A$ be a reduced irreducible divisor with isolated ordinary double points, finite stabilizer, and $K = \delta_D$. Write $g = \dim A$, let s be the number of nodes, and let $\ell = \deg(\Lambda_D)$ be the Gauss degree of the conormal variety of D.*

(a) *If g is even, then $\mathrm{cc}(K) = \Lambda_D$ is irreducible. Hence G_2, F_4, E_8 are impossible, while E_7 is forced into the 56-dimensional representation.*

(b) *If g is odd and $G^*_{D,\omega}$ is in the low one-nonzero-orbit range, then*

H, V	$\ell + s$
$G_2, 7$	7
$F_4, 26$	26
$E_7, 133$	133
$E_8, 248$	248.

If, moreover, the point-conormal summand is the zero-weight microlocal summand, then the exact node counts are

H, V	s	ℓ
$G_2, 7$	1	6
$F_4, 26$	2	24
$E_7, 133$	7	126
$E_8, 248$	8	240.

(c) *If g is odd, $\mathrm{Stab}(D) = 0$, and no two nodes differ by a torsion point, then δ_D is essentially multiplicity-free. In this torsion-separated case F_4 and E_8 cannot occur at all, E_7 cannot occur in odd ambient dimension, and G_2 is forced into its 7-dimensional representation with $\ell + s = 7$.*

The final sieve comes from invariant tensors. A full Tannaka invariant in $V_X^{\otimes n}$ is equivalent to a punctual summand in the n-fold convolution of δ_X, hence to a geometric constraint on the addition map $X^n \to A$.

Theorem 1.5 (Invariant-tensor test)**.** *Let $\mathscr{G} = G_{X,\omega}$ and $V = V_X$. If*

$$\mathrm{Hom}_{\mathscr{G}}(\mathbf{1}, V^{\otimes n}) \neq 0,$$

then, after translation, a skyscraper perverse sheaf occurs as a direct summand of a perverse cohomology sheaf of

$$Ra_{n,*}\mathrm{IC}_{X^n}[n \dim X], \qquad a_n(x_1, \dots, x_n) = x_1 + \cdots + x_n.$$

For G_2 on 7, F_4 on 26, and the adjoint representations of E_7 and E_8, the derived group has a cubic invariant. This gives a degree-three addition-map test precisely when the full Tannaka central character is trivial in degree three, and otherwise gives a degree-six test after squaring. For E_7 on 56, the quartic invariant gives a degree-four test.

These results do not settle the unrestricted singular problem. They replace it by a more concrete microlocal problem: a putative singular example must produce a clean characteristic cycle whose polar degrees and zero-weight summands match one of a very small number of exceptional Weyl-orbit patterns, and its low convolution powers must contain the punctual summands forced by the invariant theory.

2. Tannaka and microlocal conventions

Let $\mathcal{M}(A)$ denote the semisimple quotient of the category of holonomic $\mathscr{D}_A$-modules, or equivalently perverse sheaves via Riemann–Hilbert, by negligible objects. Negligible objects are those all of whose simple constituents are invariant under translation by a positive-dimensional abelian subvariety. Convolution descends to a rigid tensor product on $\mathcal{M}(A)$, and the tensor subcategory generated by any object is neutral Tannakian [10, 8].

For an integral subvariety $X \subset A$ we use the notation

$$\delta_X = \mathrm{IC}_X[\dim X].$$

The object is negligible precisely in the degenerate translation-invariant cases; throughout the main statements we exclude this case. We say that X is nondivisible if it is not stable under translation by any nonzero point of A. In the smooth nondivisible setup, the connected component of the Tannaka group acts irreducibly on $V_X = \omega(\delta_X)$ [9]. For singular X one should either impose irreducibility of the $G^*_{X,\omega}$-module V_X or apply the statements below to irreducible constituents.

2.1. **Clean cycles.** Let K be a clean semisimple object of $\mathcal{M}(A)$. Its characteristic cycle is a conic Lagrangian cycle in T^*A. Since

$$T^*A \simeq A \times H^0(A, \Omega^1_A),$$

each irreducible conic Lagrangian component has a Gauss map to the vector space $H^0(A, \Omega^1_A)$. Components with non-generically-finite Gauss map have degree zero and are called degenerate. The clean characteristic cycle

$$\mathrm{cc}(K) = \sum_i n_i \Lambda_i$$

is obtained by retaining only the nondegenerate components. Its degree is

$$\deg \mathrm{cc}(K) = \sum_i n_i \deg(\Lambda_i).$$

The index formula of Franecki–Kapranov gives

$$\deg \mathrm{cc}(K) = \chi(A, K),$$

and generic vanishing identifies this Euler characteristic with the dimension of the fiber functor:

$$\chi(A, K) = \dim \omega(K).$$

Krämer's microlocalization theorem constructs a bridge between the representation ring of the Tannaka group and a ring of clean conic Lagrangian cycles. In the form used here, it says that Weyl-group orbits of weights of the universal cover of $G^*_{K,\omega}$ are detected by effective clean cycles, and clean degree corresponds to vector-space dimension [7, 8].

3. The irreducible clean-cycle obstruction

Theorem 3.1 (Krämer's minuscule criterion)**.** *Let K be a clean simple object of $\mathcal{M}(A)$ whose Tannaka representation is irreducible for the connected derived group G^*. If $\mathrm{cc}(K)$ is irreducible, then the representation of G^* on $\omega(K)$ is minuscule.*

Proof. This is the minuscule criterion arising from Krämer's microlocalization functor: an irreducible clean characteristic cycle forces the Weyl group to act transitively on the weights of the representation, which is precisely the minuscule condition for an irreducible representation; see [7, Cor. 1.10]. □

Corollary 3.2. *Let $X \subset A$ be integral and non-negligible, and suppose that V_X is irreducible for $G^*_{X,\omega}$. If $\mathrm{cc}(\delta_X)$ is irreducible and $G^*_{X,\omega}$ is simple exceptional, then the only possible exceptional types are E_6 in the 27 or $27^\vee$ and E_7 in the 56. In particular, G_2, F_4, E_8 do not occur.*

Proof. The nontrivial minuscule representations of exceptional simple groups are exactly the 27 and $27^\vee$ of E_6 and the 56 of E_7. The types G_2, F_4, E_8 have no nontrivial minuscule representations. □

Corollary 3.3 (Smooth high-codimension comparison)**.** *Under the smoothness, nondivisibility, positivity and high-codimension hypotheses of Krämer–Lehn–Maculan, the only exceptional Tannaka group is the E_6 coming from the Fano surface of lines on a smooth cubic threefold. Hence none of G_2, F_4, E_7, E_8 occurs in that range.*

Proof. For smooth subvarieties the characteristic cycle is the conormal cycle, so the minuscule obstruction applies. Krämer–Lehn–Maculan prove that in the stated range the E_6 Fano-surface construction is the only exceptional case [9]. □

4. One nonzero orbit and zero-weight arithmetic

Let H be a connected simple algebraic group with maximal torus T and Weyl group W. For an irreducible H-module V, write

$$V = \bigoplus_{\lambda \in X^*(T)} V_\lambda, \qquad z(V) = \dim V_0.$$

Definition 4.1. We say that V is *one-nonzero-orbit* if the set of nonzero weights of V is a single W-orbit. The zero weight may or may not occur, and may occur with multiplicity.

Theorem 4.2 (One-nonzero-orbit classification)**.** *Let H be simple exceptional and let V be a nontrivial irreducible H-module. If V is one-nonzero-orbit, then V is minuscule or quasi-minuscule. The exceptional possibilities are*

H	V	*nonzero orbit size*	$z(V)$
G_2	7	6	1
F_4	26	24	2
E_6	$27, 27^\vee$	27	0
E_6	78	72	6
E_7	56	56	0
E_7	133	126	7
E_8	248	240	8.

Proof. If the zero weight is absent, the weights form one Weyl orbit and V is minuscule. If the zero weight is present and all nonzero weights form one orbit, V is quasi-minuscule. The classification of minuscule and quasi-minuscule highest weights gives the displayed list; see Bourbaki [2, Ch. VIII, §7] or Humphreys [5, §13.4]. In nonsimply laced type, the quasi-minuscule module is the little-adjoint representation: dimension 7 for G_2 and 26 for F_4. In simply laced exceptional type, the quasi-minuscule module is the adjoint representation. □

Corollary 4.3. *If $G^*_{X,\omega}$ has type G_2, F_4, E_7 or E_8 and V_X is one-nonzero-orbit, then the only possible pairs are*

$$(G_2, 7), \quad (F_4, 26), \quad (E_7, 56), \quad (E_7, 133), \quad (E_8, 248).$$

The corresponding degree patterns are

$$6 + 1, \quad 24 + 2, \quad 56, \quad 126 + 7, \quad 240 + 8.$$

4.1. **Orbit divisibility.** The following congruence is often useful when the characteristic cycle has several nonzero orbits.

Theorem 4.4 (Zero-weight congruence)**.** *Let V be an irreducible representation of a simple exceptional group H. Then*

$$\dim V - z(V)$$

is divisible by the integer m_H in the table

H	G_2	F_4	E_6	E_7	E_8
m_H	6	24	9	2	240.

Consequently, if the microlocal cycle attached to K has a zero-weight summand of clean degree $z_\mu(K)$, then

$$\chi(A, K) - z_\mu(K) \equiv 0 \pmod{m_H}.$$

Proof. The nonzero weights of V decompose into Weyl-group orbits, and weight multiplicity is constant on each orbit. Choose a nonzero dominant representative λ of such an orbit. Its stabilizer in W is a proper parabolic subgroup W_I, generated by the simple reflections whose walls contain λ. Hence the orbit size is the proper parabolic index

$$|W|/|W_I|.$$

The greatest common divisor of all proper parabolic indices is the integer m_H in the table; this standard Weyl-group calculation is recorded in Appendix A. Therefore every nonzero weight orbit contributes a multiple of m_H to $\dim V - z(V)$.

For the microlocal statement, Krämer's cycle map identifies clean degree with representation dimension and decomposes the clean cycle according to Weyl orbits of weights. Removing the zero-weight summand leaves a sum of nonzero orbit summands, hence a degree divisible by m_H. □

Remark 4.5. For E_7 the congruence is only parity, because all nonzero orbits have even size. The minimal nonzero orbit size is much stronger in small-degree arguments: it is 56 for E_7, whereas $m_{E_7} = 2$.

5. Self-duality and parity

For the four target types G_2, F_4, E_7, E_8, the longest Weyl element acts by -1. Thus every irreducible representation of the simply connected cover is self-dual. This has a geometric consequence.

Proposition 5.1 (Symmetry up to translation)**.** *Assume that $G^*_{X,\omega}$ has type G_2, F_4, E_7 or E_8, and that V_X is irreducible for $G^*_{X,\omega}$. Then*

$$X = -X + a$$

for some $a \in A$. After translating X, one may assume $X = -X$.

Proof. Since $w_0 = -1$, the G^*-module V_X is self-dual. On the geometric side, Verdier duality and inversion on A identify the dual of δ_X with δ_{-X}. Passing from the connected derived group to the full Tannaka group may twist by an invertible object, and invertible objects in the convolution category are skyscraper sheaves at points. Thus $\delta_X^\vee$ is a translate of δ_{-X}, forcing $X = -X + a$. Since multiplication by 2 is surjective on a complex abelian variety, a further translation makes X symmetric. □

Once X is symmetric, the sign of the self-duality pairing is governed by the shift in the intersection complex.

Proposition 5.2 (Parity filter)**.** *Assume $X = -X$. The Verdier-duality pairing on*

$$\delta_X = \mathrm{IC}_X[\dim X]$$

has sign $(-1)^{\dim X}$. Therefore the low one-nonzero-orbit representations obey

H, V	*invariant form on* V	$\dim X \pmod 2$
$G_2, 7$	*orthogonal*	0
$F_4, 26$	*orthogonal*	0
$E_7, 56$	*symplectic*	1
$E_7, 133$	*orthogonal*	0
$E_8, 248$	*orthogonal*	0.

Proof. The sign $(-1)^{\dim X}$ is the standard parity sign for the shifted intersection complex under the symmetric monoidal commutativity constraint for convolution; this is the parity principle used in [9, Section 8]. The 7 of G_2, the 26 of F_4, and the adjoint representations of E_7 and E_8 are orthogonal. The minuscule 56 of E_7 is symplectic. Matching the sign of the geometric pairing with the invariant form gives the table. □

6. Euler gaps

Since $\dim V_X = \chi(A, \delta_X)$, small Euler characteristic can force the representation.

Proposition 6.1 (Small Euler characteristic)**.** *Let $K = \delta_X$ and suppose $V = \omega(K)$ is irreducible for a simple group H of the indicated exceptional type. If $\chi(A, K)$ is smaller than the second nontrivial irreducible dimension in the table below, then V is the smallest nontrivial representation:*

H	*smallest nontrivial irrep*	*next nontrivial irrep*
G_2	7	14
F_4	26	52
E_7	56	133
E_8	248	3875.

In particular, an E_8 example with $\chi(A, K) < 3875$ must use the adjoint representation of dimension 248*.*

Proof. This is an immediate consequence of the equality $\dim V = \chi(A, K)$ and the low-dimensional representation tables for exceptional groups, or equivalently the Weyl dimension formula. The relevant initial dimensions are recorded in Appendix B. □

For E_7 it is sometimes useful to separate the minuscule and adjoint cases.

Corollary 6.2. *If G^* has type E_7, V is not the* 56*, and $\chi(A, K) < 912$, then V is the adjoint representation of dimension* 133*.*

7. Nodal divisors

Let $D \subset A$ be a reduced irreducible divisor whose singular locus consists of isolated ordinary double points

$$p_1, \dots, p_s.$$

Put $g = \dim A$, so $\dim D = g - 1$, and let $\Lambda_D \subset T^*A$ denote the conormal cycle of D. We assume that $\mathrm{Stab}(D)$ is finite, equivalently that the Gauss map of Λ_D is generically finite.

Theorem 7.1 (Krämer's nodal formula)**.** *For $K = \delta_D$, one has*

$$\mathrm{CC}(K) = \begin{cases} \Lambda_D, & g \text{ even}, \\ \Lambda_D + \sum_{i=1}^{s} \Lambda_{p_i}, & g \text{ odd}, \end{cases}$$

where $\Lambda_{p_i} = T^*_{p_i} A$ has Gauss degree 1.

Proof. This is [8, Lemma 6.5.2]. Locally, an ordinary double point has Milnor number one, and Picard–Lefschetz monodromy acts on the one-dimensional vanishing cohomology by $(-1)^g$, giving the displayed parity. □

7.1. **Unconditional nodal constraints.** The following theorem uses only the nodal formula, Euler gaps and parity. It does not identify the point-conormal part with the zero-weight summand.

Theorem 7.2 (Unconditional low-range nodal constraints)**.** *Let $D \subset A$ be as above, put $K = \delta_D$, and write*

$$\ell = \deg(\Lambda_D) > 0.$$

(i) *If $G^*_{D,\omega}$ has type G_2 and $\chi(A, K) < 14$, then $V = 7$, g is odd, and*

$$\ell + s = 7.$$

(ii) *If $G^*_{D,\omega}$ has type F_4 and $\chi(A, K) < 52$, then $V = 26$, g is odd, and*

$$\ell + s = 26.$$

(iii) *If $G^*_{D,\omega}$ has type E_8 and $\chi(A, K) < 3875$, then $V = 248$, g is odd, and*

$$\ell + s = 248.$$

(iv) *If $G^*_{D,\omega}$ has type E_7, $V \neq 56$, and $\chi(A, K) < 912$, then $V = 133$, g is odd, and*

$$\ell + s = 133.$$

(v) *If $G^*_{D,\omega}$ has type E_7 and $V = 56$, then g is even and*

$$\ell = 56.$$

Proof. The Euler gap Proposition 6.1 and Corollary 6.2 identify the representations. The parity filter says that $\dim D = g - 1$ is even for G_2 on 7, F_4 on 26, adjoint E_7, and adjoint E_8; hence g is odd in (i)–(iv). Krämer's nodal formula gives

$$\chi(A, K) = \deg \mathrm{CC}(K) = \ell + s.$$

For E_7 on 56, the invariant form is symplectic, so $\dim D$ is odd and g is even; the nodal formula gives $\mathrm{CC}(K) = \Lambda_D$ and $\ell = 56$. □

7.2. **Zero-separated nodal cycles.** The exact node counts predicted by the one-nonzero-orbit table require an additional microlocal separation hypothesis.

Definition 7.3 (Zero-separated nodal cycle)**.** Assume g is odd and V is one-nonzero-orbit. The nodal characteristic cycle is called *zero-separated* if the point-conormal summand

$$\sum_{i=1}^{s} \Lambda_{p_i}$$

is exactly the clean cycle attached to the zero-weight orbit, and Λ_D is the clean cycle attached to the unique nonzero-weight orbit.

Theorem 7.4 (Zero-separated nodal rigidity)**.** *Let $D \subset A$ be a nodal divisor satisfying the hypotheses of Theorem 7.2. In the low one-nonzero-orbit cases, assume moreover that the nodal characteristic cycle is zero-separated. Then*

H, V	s	ℓ	$\chi(A, \delta_D)$
$G_2, 7$	1	6	7
$F_4, 26$	2	24	26
$E_7, 133$	7	126	133
$E_8, 248$	8	240	248.

Proof. By Theorem 4.2, the zero-weight multiplicities in the four rows are $1, 2, 7, 8$, and the nonzero orbit sizes are $6, 24, 126, 240$. Under zero-separation, the zero-weight clean degree is the number s of point-conormal components, since each point-conormal has degree 1. The remaining degree is ℓ. □

7.3. **The torsion-separated case.** Krämer also gives a strong multiplicity-free criterion for nodal divisors. Recall that a clean object K is *essentially multiplicity-free* if $[n]_*(\mathrm{cc}(K))$ is reduced for every multiplication map $[n] : A \to A$. In that case, the associated representation is weight-multiplicity-free in the sense of representation theory [8, Section 6.5].

Theorem 7.5 (Torsion-separated nodal sieve)**.** *Let $D \subset A$ be a reduced irreducible divisor with isolated ordinary double points and $\mathrm{Stab}(D) = 0$. If g is odd, assume in addition that no two nodes differ by a torsion point. Suppose $G^*_{D,\omega}$ is one of G_2, F_4, E_7, E_8 and $V = \omega(\delta_D)$ is irreducible for it. Then:*

(a) *F_4 and E_8 do not occur.*
(b) *If g is odd, then E_7 does not occur.*
(c) *If G_2 occurs, then g is odd, V is the 7-dimensional representation, and*

$$\deg(\Lambda_D) + s = 7.$$

(d) *If E_7 occurs, then g is even and V is the 56-dimensional representation.*

Proof. If g is even, Krämer's nodal formula gives $\mathrm{cc}(\delta_D) = \Lambda_D$, so Corollary 3.2 excludes G_2, F_4, E_8 and leaves only E_7 on 56.

Assume now that g is odd. By [8, Lemma 6.5.2], the hypotheses $\mathrm{Stab}(D) = 0$ and torsion-separation of the nodes imply that δ_D is essentially multiplicity-free. Hence V is a weight-multiplicity-free irreducible representation. Krämer's classification table for weight-multiplicity-free irreducible representations of simple groups [8, Lemma 6.5.1] contains, among the four target exceptional types, only the 7 of G_2 and the 56 of E_7.

The 56 of E_7 is symplectic, but for a divisor in odd ambient dimension one has $\dim D = g - 1$ even, so the parity filter requires an orthogonal representation. Thus E_7 is impossible for odd g. If G_2 occurs, its only weight-multiplicity-free representation is the 7, and the nodal formula gives $\deg \mathrm{CC}(\delta_D) = \ell + s = 7$. □

Remark 7.6. The torsion-separated theorem deliberately does not assert an exact number of nodes in the G_2 case. The exact count $s = 1$ follows under the zero-separation hypothesis of Definition 7.3. Without such an identification, the equality $\ell + s = 7$ is the unconditional conclusion.

8. Invariant tensors and addition maps

Let $K = \delta_X$ and let

$$\mathscr{G} = G_{X,\omega}, \qquad G^* = (\mathscr{G}^\circ)_{\mathrm{der}}, \qquad V = \omega(K).$$

For $n \geq 1$ write

$$a_n : X^n \longrightarrow A, \qquad (x_1, \ldots, x_n) \longmapsto x_1 + \cdots + x_n.$$

Proposition 8.1 (Punctual summands from full invariants)**.** *If*

$$\mathrm{Hom}_{\mathscr{G}}(\mathbf{1}, V^{\otimes n}) \neq 0,$$

then the unit object occurs as a direct summand of K^{*n} in the semisimple quotient tensor category. Equivalently, after translating the unit by an invertible skyscraper object if necessary, a skyscraper perverse sheaf occurs as a direct summand of a perverse cohomology sheaf of

$$Ra_{n,*}\mathrm{IC}_{X^n}[n \dim X].$$

Proof. Under the Tannakian equivalence, the multiplicity of the unit object in K^{*n} is exactly

$$\dim \mathrm{Hom}_{\mathscr{G}}(\mathbf{1}, V^{\otimes n}).$$

The unit for convolution is the skyscraper sheaf at the origin. Tensoring by an invertible object translates it to a skyscraper sheaf at another point of A. The displayed direct image is the geometric realization of the n-fold convolution. □

The central character of the full Tannaka group matters. An invariant for the connected derived group need not be invariant under the full group.

Corollary 8.2 (Low-degree exceptional tests)**.** *Assume X has been translated so that $K \simeq K^\vee$, and let η be the central character by which $\mathscr{G}/G^*$ acts on the irreducible G^*-module V.*

(a) *For G_2 on 7, F_4 on 26, adjoint E_7, and adjoint E_8, the connected derived group has a cubic invariant. It yields a degree-three addition-map test exactly when $\eta^3 = 1$. Since self-duality gives $\eta^2 = 1$, the square of the cubic invariant always yields a degree-six addition-map test.*
(b) *For E_7 on 56, the quartic invariant yields a degree-four addition-map test. The symplectic form gives the degree-two evaluation pairing.*

Proof. The 7 of G_2 preserves a nonzero alternating 3-form; the 26 of F_4 is the trace-free Albert-algebra representation with its cubic norm; and the adjoint representation of a simple Lie algebra has the invariant trilinear form

$$(x, y, z) \longmapsto \kappa([x, y], z),$$

where κ is the Killing form. These tensors are invariant under G^* and transform under the full group by η^3. Hence they are full invariants precisely when $\eta^3 = 1$. Since $K \simeq K^\vee$, the full group preserves a nondegenerate bilinear form on V, so $\eta^2 = 1$; the square of a cubic invariant has central character $\eta^6 = 1$.

For the 56 of E_7, the primitive invariant has degree four. Its central character is η^4, which is trivial once $\eta^2 = 1$. □

Remark 8.3. The invariant-tensor test is geometric rather than merely numerical. To exclude a candidate X, one may show that the relevant addition map $a_n : X^n \to A$ cannot have a punctual direct summand in the required perverse cohomology. In the E_6 Fano-surface example, the cubic invariant on the 27 is reflected by the special fiber of $F(Y)^3 \to \mathrm{Alb}(F(Y))$ parametrizing triples of coplanar lines on the cubic threefold.

9. A candidate-elimination protocol

The preceding results provide a practical finite sieve for proposed examples.

(1) Compute $\chi(A, \delta_X) = \deg \mathrm{cc}(\delta_X)$. Euler gaps often force the representation.
(2) Compute the clean characteristic cycle. If it is irreducible, the minuscule obstruction excludes G_2, F_4, E_8 and leaves only E_7 on 56.
(3) If the clean cycle is reducible, identify the Weyl-orbit decomposition. If all nonzero weights form one orbit, the only target possibilities are G_2 on 7, F_4 on 26, E_7 on 56 or 133, and E_8 on 248.

(4) Separate the zero-weight summand. The congruence

$$\chi(A, \delta_X) - z_\mu(\delta_X) \equiv 0 \pmod{6, 24, 2, 240}$$

for G_2, F_4, E_7, E_8 is a fast polar-degree check.
(5) Use self-duality. If X is not symmetric up to translation, none of the four target types occurs. If it is symmetric, the invariant form fixes $\dim X \pmod 2$ in the low representations.
(6) For nodal divisors, apply Theorem 7.1. In even ambient dimension the clean cycle is irreducible. In odd ambient dimension the point-conormal contributions give the degree equations of Theorem 7.2. If the nodes are torsion-separated, Theorem 7.5 excludes F_4, E_8 and odd-dimensional E_7.
(7) Test the required invariant tensors on addition maps. Cubic invariants give degree-three or degree-six tests depending on the full central character; the quartic invariant of E_7 on 56 gives a degree-four test.

10. What remains open

The unrestricted singular problem is concentrated in the failure of zero-separation and in higher-orbit characteristic cycles. A putative example of type G_2, F_4, E_7 or E_8 must evade the following constraints simultaneously:

(1) its clean characteristic cycle must be reducible unless it is the minuscule E_7 case;
(2) its clean degrees must satisfy the zero-weight congruence;
(3) its support must be symmetric up to translation and obey the parity filter;
(4) in nodal situations it must fail the torsion-separated multiplicity-free criterion unless it lies in the small surviving cases;
(5) its low convolution powers must contain the punctual summands forced by invariant tensors.

The most concrete next target is the following.

Question 10.1 (Zero-orbit rigidity)**.** *Let $X \subset A$ be integral, non-negligible and nondivisible, and suppose $G^*_{X,\omega}$ is simple exceptional while V_X is not minuscule. Is the zero-weight microlocal summand forced by a special addition-map geometry of cubic-threefold type? In particular, can such a summand occur for the target types G_2, F_4, E_7, E_8?*

A positive answer would rule out all non-minuscule target cases. The only remaining case would then be the minuscule 56 of E_7, already excluded in the smooth high-codimension setting by Krämer–Lehn–Maculan.

Appendix A. Weyl-index certificate

Let W be the Weyl group of a simple exceptional root system. If $\lambda \neq 0$ is a dominant weight, then its stabilizer in W is the proper parabolic subgroup generated by the simple reflections s_α with

$$\langle \lambda, \alpha^\vee \rangle = 0.$$

Thus every nonzero Weyl orbit has size $|W|/|W_I|$ for a proper standard parabolic subgroup W_I. Since every proper parabolic is contained in a maximal proper parabolic, the greatest common divisor of all proper parabolic indices is the greatest common divisor of the maximal parabolic indices.

The resulting values are:

H	$\lvert W_H\rvert$	$\gcd_{I\subsetneq\Delta} \lvert W_H\rvert/\lvert W_I\rvert$	$\min_{\lambda\neq 0} \lvert W_H\lambda\rvert$
G_2	12	6	6
F_4	1152	24	24
E_6	51840	9	27
E_7	2903040	2	56
E_8	696729600	240	240.

For example, in type E_8 the smallest nonzero orbit is the root orbit of size 240, and every nonzero orbit size is divisible by 240. In type F_4, the short-root and long-root orbits both have size 24, and all nonzero orbit sizes are divisible by 24. In type G_2, the two root orbits have size 6.

Appendix B. Low-dimensional exceptional representations

The small irreducible representation dimensions used in the Euler-gap arguments are:

G_2	$7, 14, 27, 64, 77, 77', 182, \ldots$
F_4	$26, 52, 273, 324, 1053, 1274, \ldots$
E_7	$56, 133, 912, 1463, 1539, 6480, \ldots$
E_8	$248, 3875, 27000, 30380, 147250, \ldots.$

They follow from the Weyl dimension formula and agree with the standard representation tables.

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