FINITE-ÉTALE TOWER PROFILES, RAMIFICATION GRADIENTS, AND CYCLIC PRYM P-RANKS OVER $\overline{\mathbb{F}}_Q$

PRZEMEK CHOJECKI

ABSTRACT. Let $k = \overline{\mathbb{F}}_q$ and let $X, Y/k$ be smooth projective curves of genus at least 2. We study the problem of whether X and Y admit a common finite étale cover. The paper does not solve the problem. Instead, it isolates two finite-étale commensurability profiles which appear to be natural obstructions. The first is a directed ramification gradient

$$\sigma(X,Y) = \sup_{U\to X,\, h:U\to Y} \frac{\deg(h)(2g(Y)-2)}{2g(U)-2},$$

where $U \to X$ is finite étale and h is finite separable. We prove that σ is invariant under replacing either variable by a finite étale cover, and satisfies a triangle inequality. The second is a cyclic Prym p-rank profile. For a prime $\ell \neq p$ we define

$$M_\ell(X) = \inf_{X_0\to X} \sup_{U\to X_0,\, V\to U\ \mathbb{Z}/\ell\text{-étale}} \left(1 - \frac{f(P_{V/U})}{\dim P_{V/U}}\right),$$

where $P_{V/U}$ is the Prym variety and f denotes p-rank. We prove that M_ℓ is a finite-étale commensurability invariant and derive the exact update formula

$$D_p(V) = \frac{1}{\ell} D_p(U) + \frac{\ell-1}{\ell}\delta_p(P_{V/U}).$$

We also record why simpler invariants, such as ordinary/nonordinary status and bare normalized ramification, are probably insufficient. The final sections formulate a positive "virtual orbifold uniformization" route and a negative "Prym gap" route.

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1. The question

Throughout, $k = \overline{\mathbb{F}}_q$ is the algebraic closure of a finite field of characteristic $p > 0$. Curves are assumed smooth, projective, geometrically connected over k, and of genus at least 2, unless explicitly stated otherwise.

Question 1.1 (Common finite-étale cover problem)**.** *Given two curves $X, Y/k$, does there exist a curve Z/k and finite étale morphisms*

$$Z \longrightarrow X, \qquad Z \longrightarrow Y?$$

Equivalently, are all such curves mutually isogenous in the sense of admitting a bi-finite-étale correspondence? The affine analogue is known to have a positive answer by Tamagawa: any two affine smooth connected curves over $\overline{\mathbb{F}}_q$ are isogenous in this sense [9]. The proper case is precisely where boundary ramification can no longer be hidden at the punctures.

We write

$$X \sim_{\text{ét}} Y$$

if X and Y have a common finite étale cover. The purpose of this note is to construct invariants of the equivalence relation generated by finite étale covers, and to formulate concrete ways in which these invariants might either obstruct or support a positive answer to Question 1.1.

Remark 1.2. The finite-étale cover relation may be expressed group-theoretically. If $X \sim_{\text{ét}} Y$, then the profinite groups $\pi_1^{\text{ét}}(X)$ and $\pi_1^{\text{ét}}(Y)$ have isomorphic open subgroups. Thus any profinite commensurability invariant of $\pi_1^{\text{ét}}(X)$ gives an obstruction. The difficulty is to find a computable virtual invariant which is not automatically universal over $\overline{\mathbb{F}}_q$.

2. Finite-étale tower profiles

This section records the basic cofinality mechanism used repeatedly below.

Definition 2.1. Let $\mathcal{E}(X)$ denote the directed system of connected finite étale covers $U \to X$. A *tail property* of X is a construction made from $\mathcal{E}(X)$ which is unchanged after replacing $\mathcal{E}(X)$ by a cofinal subsystem.

Lemma 2.2 (Cofinality)**.** *Let $X' \to X$ be a connected finite étale cover. Then the connected finite étale covers of X' form a cofinal subsystem of the connected finite étale covers of X: for every $U \to X$ in $\mathcal{E}(X)$, every connected component of*

$$U \times_X X'$$

is finite étale over both U and X'.

Proof. Finite étaleness is stable under base change and composition. The fiber product is finite étale over U and over X'. Since k is algebraically closed, a connected component is again a smooth projective connected curve. This component dominates both U and X', giving cofinality. □

Definition 2.3 (Lower and upper tail envelopes)**.** Let a be a real-valued invariant of curves, with values in a compact interval. Define

$$\underline{a}(X) = \sup_{X_0 \to X} \inf_{U \to X_0} a(U),$$

$$\overline{a}(X) = \inf_{X_0 \to X} \sup_{U \to X_0} a(U),$$

where all covers are connected finite étale.

Proposition 2.4. *The quantities $\underline{a}(X)$ and $\overline{a}(X)$ are invariant under replacing X by a connected finite étale cover.*

Proof. This is a direct consequence of Lemma 2.2. Passing from X to X' only discards a non-cofinal initial portion of the same directed tower; the subsequent tails are cofinal in one another. Since sup inf and inf sup only depend on eventual tails, the values are unchanged. □

3. A ramification gradient

For a curve C set

$$\kappa(C) = 2g(C) - 2.$$

Thus $\kappa(C) > 0$ for all curves under consideration. Let $h : C \to D$ be a finite separable morphism of smooth projective curves. By Riemann-Hurwitz,

$$\kappa(C) = \deg(h)\kappa(D) + \deg \mathrm{Diff}_{C/D}\,.$$

Definition 3.1 (Ramification score and defect)**.** For curves $X, Y/k$ define

$$\sigma(X,Y) = \sup_{(U,h)} \frac{\deg(h)\kappa(Y)}{\kappa(U)},$$

where $U \to X$ ranges over connected finite étale covers and $h : U \to Y$ ranges over finite separable morphisms. If the indexing set is empty, set $\sigma(X,Y) = 0$. Define also

$$\varepsilon(X,Y) = 1 - \sigma(X,Y).$$

Equivalently,

$$\varepsilon(X,Y) = \inf_{(U,h)} \frac{\deg \mathrm{Diff}_{U/Y}}{\kappa(U)}.$$

The invariant $\sigma(X,Y)$ measures how close one can get to a finite étale map from a finite étale cover of X to Y.

Proposition 3.2. *If $X' \to X$ and $Y' \to Y$ are connected finite étale covers, then*

$$\sigma(X',Y) = \sigma(X,Y), \qquad \sigma(X,Y') = \sigma(X,Y).$$

Consequently $\varepsilon(X,Y)$ is invariant under replacing either variable by a finite étale cover.

Proof. First consider $X' \to X$. Every finite étale cover of X' is also a finite étale cover of X, so $\sigma(X',Y) \le \sigma(X,Y)$. Conversely, let $U \to X$ be finite étale and let $h : U \to Y$ be finite separable. Choose a connected component V of $U \times_X X'$. If $m = \deg(V/U)$, then $V \to X'$ is finite étale,

$$\kappa(V) = m\kappa(U), \qquad \deg(V \to Y) = m \deg(h),$$

and hence the score is unchanged:

$$\frac{\deg(V \to Y)\kappa(Y)}{\kappa(V)} = \frac{\deg(h)\kappa(Y)}{\kappa(U)}.$$

Thus $\sigma(X',Y) \ge \sigma(X,Y)$.

Now consider $Y' \to Y$ of degree n. A map $U \to Y'$ composed with $Y' \to Y$ has the same score because $\kappa(Y') = n\kappa(Y)$. Hence $\sigma(X,Y) \ge \sigma(X,Y')$. Conversely, given $h : U \to Y$, take a connected component V of $U \times_Y Y'$. Since $Y' \to Y$ is finite étale, $V \to U$ is finite étale; write $m = \deg(V/U)$. Then

$$\kappa(V) = m\kappa(U),$$

and, from the equality of degrees over Y,

$$\deg(V \to Y') = \frac{m \deg(h)}{n}.$$

Therefore

$$\frac{\deg(V \to Y')\kappa(Y')}{\kappa(V)} = \frac{(m \deg(h)/n)n\kappa(Y)}{m\kappa(U)} = \frac{\deg(h)\kappa(Y)}{\kappa(U)}.$$

Thus $\sigma(X, Y') \geq \sigma(X, Y)$. □

Proposition 3.3 (Triangle inequality)**.** *For any curves $X, Y, Z/k$,*

$$\sigma(X, Z) \geq \sigma(X, Y)\sigma(Y, Z).$$

Equivalently, $d_{\mathrm{ram}}(X, Y) = -\log \sigma(X, Y)$, with $-\log(0) = +\infty$, is a directed pseudo-distance on finite-étale commensurability classes.

Proof. Choose data

$$U \to X, \qquad h : U \to Y,$$

with score close to $\sigma(X, Y)$, and data

$$V \to Y, \qquad \ell_V : V \to Z,$$

where $V \to Y$ is finite étale and the score is close to $\sigma(Y, Z)$. Let $d = \deg(h)$, $m = \deg(V/Y)$, and $r = \deg(\ell_V)$. Let W be a connected component of $U \times_Y V$, and write $e = \deg(W/U)$. Then $W \to U$ is finite étale, so $W \to X$ is finite étale, and

$$\kappa(W) = e\kappa(U).$$

Moreover

$$\deg(W \to V) = \frac{ed}{m}.$$

The composed map $W \to V \to Z$ therefore has degree $r(ed/m)$. Its score is

$$\frac{r(ed/m)\kappa(Z)}{e\kappa(U)} = \left(\frac{d\kappa(Y)}{\kappa(U)}\right)\left(\frac{r\kappa(Z)}{m\kappa(Y)}\right).$$

Taking suprema gives the result. □

Corollary 3.4. *If $X \sim_{\text{ét}} Y$, then*

$$\sigma(X, Y) = \sigma(Y, X) = 1.$$

Consequently, if $\sigma(X, Y) < 1$ or $\sigma(Y, X) < 1$, then X and Y do not have a common finite étale cover.

Proof. If $Z \to X$ and $Z \to Y$ are finite étale, then the datum $U = Z$ and $h : Z \to Y$ gives score 1, because

$$\kappa(Z) = \deg(Z/Y)\kappa(Y).$$

The same argument in the other direction gives $\sigma(Y, X) = 1$. □

Corollary 3.5 (Attainment criterion)**.** *Suppose $\sigma(X, Y) = 1$ is attained by some finite étale $U \to X$ and finite separable $h : U \to Y$. Then X and Y have a common finite étale cover.*

Proof. If the score is 1, then Riemann-Hurwitz gives $\deg \mathrm{Diff}_{U/Y} = 0$. Thus h is unramified. Since h is finite separable between smooth projective curves, it is finite étale. Hence U is a common finite étale cover. □

Problem 3.6 (Ramification gap)**.** *Find curves $X, Y/k$ such that*

$$\sigma(X, Y) < 1.$$

Equivalently, prove that there is a constant $c(X, Y) > 0$ such that every finite étale $U \to X$ and finite separable $h : U \to Y$ satisfy

$$\deg \mathrm{Diff}_{U/Y} \geq c(X, Y)\kappa(U).$$

4. Why bare ramification is probably too weak

The ramification gradient is attractive because it directly targets the common-cover question. However, ramification alone is too flexible if the source curve is allowed to vary freely.

Proposition 4.1 (High degree covers with bounded ramification)**.** *Let Y/k be a smooth projective curve of genus at least 1. Fix a prime $\ell \neq p$. There exist finite separable covers*

$$C_m \longrightarrow Y$$

of degrees tending to infinity such that

$$\deg \mathrm{Diff}_{C_m/Y} \leq 2(\ell - 1)$$

for all m in an infinite set.

Proof. Choose $m \geq 2$ prime to p and let $G_m = (\mathbb{Z}/\ell)^m \rtimes \mathbb{Z}/m$, where $\mathbb{Z}/m$ cyclically permutes the m factors. Let G_m act on m blocks, each of size ℓ: the normal subgroup $(\mathbb{Z}/\ell)^m$ acts by translations inside the blocks and $\mathbb{Z}/m$ cyclically permutes the blocks. Let s be a block rotation and let v be an ℓ-cycle on one block. Then the commutator

$$c = [s, v]$$

acts as an ℓ-cycle on one block, the inverse ℓ-cycle on the next block, and fixes all other points. Thus the tame ramification contribution of c in this permutation representation is exactly

$$2(\ell - 1).$$

Let $y \in Y(k)$. The prime-to-p tame fundamental group of $Y \setminus \{y\}$ admits generators a_i, b_i and an inertia generator τ with relation

$$\prod_{i=1}^{g(Y)} [a_i, b_i]\tau = 1.$$

Send $a_1 \mapsto s$, $b_1 \mapsto v$, $\tau \mapsto c^{-1}$, and all other generators to 1. This gives a connected tame cover of $Y \setminus \{y\}$ because the induced permutation group is transitive. Its smooth compactification $C_m \to Y$ is branched only above y, with tame local monodromy c^{-1}. Therefore

$$\deg \mathrm{Diff}_{C_m/Y} = 2(\ell - 1),$$

while the degree of $C_m \to Y$ is $m\ell$, which tends to infinity. □

Corollary 4.2. *For a fixed target Y, there exist finite separable covers $C_m \to Y$ with*

$$\frac{\deg \mathrm{Diff}_{C_m/Y}}{\kappa(C_m)} \longrightarrow 0.$$

Proof. By Riemann-Hurwitz,

$$\kappa(C_m) = \deg(C_m/Y)\kappa(Y) + O(1).$$

The numerator is bounded and the degree tends to infinity. □

Warning 4.3. Proposition 4.1 does not say anything directly about $\sigma(X, Y)$, because the curves C_m are not required to lie in the finite-étale tower of a fixed source X. Rather, it shows that a negative proof cannot rely only on the existence of high-degree nearly unramified covers of Y; it must also use structure of the tower of X.

5. Hasse-Witt defect and its limitations

Let A/k be an abelian variety. Write $f(A)$ for its p-rank, so

$$\#A[p](k) = p^{f(A)}.$$

For a curve C, write

$$f(C) = f(\mathrm{Jac}(C)).$$

Equivalently, $f(C) = \dim_{\mathbb{F}_p} \mathrm{Hom}(\pi_1^{\text{ét}}(C), \mathbb{Z}/p)$.

Definition 5.1 (Hasse-Witt defect)**.** For a curve C define

$$D_p(C) = 1 - \frac{f(C)-1}{g(C)-1} = \frac{g(C)-f(C)}{g(C)-1}.$$

Thus $D_p(C) = 0$ if and only if C is ordinary.

The tail envelopes $\underline{D}_p(X)$ and $\overline{D}_p(X)$ of Definition 2.3 are finite-étale commensurability invariants by Proposition 2.4. The following proposition records a standard consequence of Tamagawa's p-average formula, in the form generalized by Yang.

Proposition 5.2 (Tamagawa-Yang consequence)**.** *Assume the usual p-average formula for the maximal prime-to-p abelian covers of smooth proper curves: for every curve C/k there exists a sequence of connected finite étale prime-to-p abelian covers $C_n \to C$ such that*

$$\frac{f(C_n)-1}{\deg(C_n/C)} \longrightarrow g(C) - 1.$$

Then

$$\underline{D}_p(X) = 0$$

for every curve X/k.

Proof. Let $X_0 \to X$ be any connected finite étale cover. Applying the stated p-average consequence to $C = X_0$, we obtain covers $C_n \to X_0$ with

$$\frac{f(C_n)-1}{\deg(C_n/X_0)} \longrightarrow g(X_0) - 1.$$

Riemann-Hurwitz for the finite étale cover $C_n \to X_0$ gives

$$g(C_n) - 1 = \deg(C_n/X_0)(g(X_0) - 1).$$

Therefore

$$\frac{f(C_n)-1}{g(C_n)-1} \longrightarrow 1,$$

so $D_p(C_n) \to 0$. Since this can be done after every $X_0 \to X$, every tail of the finite-étale tower contains covers of arbitrarily small Hasse-Witt defect. Hence $\underline{D}_p(X) = 0$. □

Remark 5.3. The precise p-average formalism is due to Tamagawa and was generalized by Yang to pointed stable curves; see [8, 10]. Proposition 5.2 uses only the displayed consequence. If one wants to avoid importing the full formalism, the proposition may be read as conditional on that consequence.

Proposition 5.4 (Ordinarity is not a virtual invariant)**.** *The binary property "ordinary" does not define a useful finite-étale commensurability invariant.*

Proof. Raynaud proved that every smooth proper connected curve of genus at least 2 in characteristic $p > 0$ admits a finite solvable prime-to-p Galois étale cover which is not ordinary [6]. Moreover, once Y is nonordinary, every further finite étale cover $Z \to Y$ is nonordinary. Indeed, pullback and norm show that $\mathrm{Jac}(Y)$ is an isogeny factor of $\mathrm{Jac}(Z)$. If Z were ordinary, then every isogeny factor of $\mathrm{Jac}(Z)$ would be ordinary, contradiction. □

Thus the ordinary/nonordinary dichotomy fails in both directions: every tower eventually contains a permanently nonordinary region, while Proposition 5.2 says that every tail contains covers that are asymptotically ordinary in normalized p-rank.

6. Cyclic Prym defect

The preceding section suggests that one should not look only at the p-rank of curves in the tower. One should instead look at the new parts produced by controlled finite étale steps.

Fix a prime $\ell \neq p$. Let

$$\pi : V \longrightarrow U$$

be a connected cyclic étale cover of degree ℓ. Let

$$P_\pi = \ker\big(\mathrm{Nm}_\pi : \mathrm{Jac}(V) \to \mathrm{Jac}(U)\big)^0$$

be the Prym variety. Then

$$\dim P_\pi = (\ell - 1)(g(U) - 1),$$

and, up to isogeny,

$$\mathrm{Jac}(V) \sim \mathrm{Jac}(U) \times P_\pi.$$

Definition 6.1 (Prym p-defect)**.** For such a cyclic étale cover define

$$\delta_p(P_\pi) = 1 - \frac{f(P_\pi)}{\dim P_\pi}.$$

Thus $\delta_p(P_\pi) = 0$ if and only if the Prym is ordinary, while $\delta_p(P_\pi) = 1$ if and only if the Prym has p-rank zero.

Definition 6.2 (Eventual upper cyclic Prym defect)**.** Define

$$M_\ell(X) = \inf_{X_0 \to X} \sup_{\substack{U \to X_0 \\ \pi: V \to U \ \mathbb{Z}/\ell\text{-étale}}} \delta_p(P_\pi),$$

where all curves are connected and all covers are finite étale.

Proposition 6.3. *For every prime $\ell \neq p$, the quantity $M_\ell(X)$ is invariant under replacing X by a connected finite étale cover. Hence*

$$X \sim_{\text{ét}} Y \quad \Longrightarrow \quad M_\ell(X) = M_\ell(Y)$$

for all $\ell \neq p$.

Proof. This is the same cofinality argument as Proposition 2.4. Replacing X by a finite étale cover X' replaces the tower $\mathcal{E}(X)$ by a cofinal subsystem. The outer $\inf_{X_0 \to X}$ exactly removes the initial portion of the tower, so the value is unchanged. □

Theorem 6.4 (Exact defect update formula)**.** *Let $\pi : V \to U$ be a connected cyclic étale cover of degree $\ell \neq p$. Then*

$$D_p(V) = \frac{1}{\ell} D_p(U) + \frac{\ell - 1}{\ell} \delta_p(P_\pi).$$

Proof. Riemann-Hurwitz gives

$$g(V) - 1 = \ell(g(U) - 1).$$

The isogeny

$$\mathrm{Jac}(V) \sim \mathrm{Jac}(U) \times P_\pi$$

implies additivity of p-rank:

$$f(V) = f(U) + f(P_\pi).$$

Therefore

$$g(V) - f(V) = \big(g(U) - f(U)\big) + \big(\dim P_\pi - f(P_\pi)\big).$$

Divide by $g(V) - 1 = \ell(g(U) - 1)$ and use

$$\dim P_\pi = (\ell - 1)(g(U) - 1).$$

This yields the asserted formula. □

Corollary 6.5 (Double covers)**.** *Assume $p \neq 2$ and let $\pi : V \to U$ be an unramified double cover. Then*

$$D_p(V) = \frac{1}{2} D_p(U) + \frac{1}{2} \delta_p(P_\pi).$$

In particular, ordinary Pryms push D_p downward by a factor of 2, while p-rank-zero Pryms push D_p toward 1.

Corollary 6.6 (A Prym obstruction)**.** *If there exist curves $X, Y/k$ and a prime $\ell \neq p$ such that*

$$M_\ell(X) \neq M_\ell(Y),$$

then X and Y do not have a common finite étale cover.

7. The negative route: Prym gaps

The invariant M_ℓ suggests a concrete obstruction.

Conjecture 7.1 (Prym gap for a tower)**.** *There exist a curve X/k, a prime $\ell \neq p$, and a constant $c > 0$ such that for every connected finite étale cover $U \to X$ and every connected cyclic étale cover $\pi : V \to U$ of degree ℓ,*

$$f(P_\pi) \geq c \cdot \dim P_\pi.$$

Equivalently,

$$\delta_p(P_\pi) \leq 1 - c$$

uniformly throughout the finite-étale tower of X.

Proposition 7.2. *If Conjecture 7.1 holds for X and ℓ, then*

$$M_\ell(X) \leq 1 - c < 1.$$

Consequently, if there is another curve Y with $M_\ell(Y) = 1$, then X and Y do not have a common finite étale cover.

Proof. The displayed inequality on $f(P_\pi)$ is exactly the assertion that every term in every inner supremum defining $M_\ell(X)$ is at most $1 - c$. Hence the infimum over tails is also at most $1 - c$. The final assertion follows from Corollary 6.6. □

Raynaud-Tamagawa theta divisors appear to be the natural geometric objects behind Conjecture 7.1. A nontrivial ℓ-torsion line bundle $L \in \mathrm{Jac}(U)[\ell]$ determines a cyclic étale cover $V_L \to U$. The new Hasse-Witt contribution of V_L/U is controlled by generalized Hasse-Witt invariants; degeneracy is detected by theta-type loci in the ℓ-torsion. This is precisely the setting of Raynaud, Tamagawa, Nakajima, and Yang [6, 4, 10, 11].

Problem 7.3 (Concrete negative test, $\ell = 2$)**.** *Assume $p \neq 2$. Find a curve X/k and $c > 0$ such that every unramified double cover $V \to U$ of every finite étale $U \to X$ satisfies*

$$f(P_{V/U}) \geq c(g(U) - 1).$$

Then $M_2(X) < 1$.

Existing Prym p-rank results show both flexibility and constraints. Ozman-Pries prove that for a generic curve of genus g and p-rank f, the Pryms of all unramified $\mathbb{Z}/\ell$-covers are ordinary, and also construct nonordinary Pryms in other regions of moduli [5]. Celik–Elias–Gunes–Newton–Ozman–Pries–Thomas construct low-p-rank Prym phenomena in several cases, including unramified double covers with supersingular Prym under congruence and small-prime hypotheses [2]. The missing ingredient for Conjecture 7.1 is not moduli-theoretic nonemptiness, but control inside the finite-étale tower of one fixed curve.

8. The positive route: virtual uniformization

The most direct positive strategy would be stronger than the common-cover problem itself.

Conjecture 8.1 (Virtual tame orbifold uniformization)**.** *There exists a fixed tame hyperbolic orbifold curve $\mathcal{O}/k$ such that for every smooth projective curve X/k of genus at least 2, there exists a connected finite étale cover $U \to X$ and a finite representable étale morphism of stacks*

$$U \longrightarrow \mathcal{O}.$$

Proposition 8.2. *Conjecture 8.1 implies a positive answer to Question 1.1.*

Proof. Let $X, Y/k$ be curves. By Conjecture 8.1, choose finite étale covers

$$U \to X, \qquad V \to Y$$

and finite étale maps $U \to \mathcal{O}$, $V \to \mathcal{O}$. A connected component of

$$U \times_{\mathcal{O}} V$$

is finite étale over both U and V, hence finite étale over both X and Y. □

Remark 8.3. A Belyi-type map to $\mathbb{P}^1$ is not enough for Conjecture 8.1. One needs, after passing to a finite étale cover of the source curve, uniform orbifold ramification compatible with a fixed tame orbifold. Passing to the Galois closure of a Belyi map usually ramifies over the original curve, so it does not produce a finite étale cover of X.

There is also a positive interpretation of the Prym invariant.

Conjecture 8.4 (Universal Prym-density)**.** *For some prime $\ell \neq p$, or for every such prime, one has*

$$M_\ell(X) = 1$$

for every curve X/k.

Remark 8.5. Conjecture 8.4 would not by itself prove the common-cover problem. It would only show that the cyclic Prym defect profile considered here is universal and therefore gives no obstruction. It would nevertheless be strong evidence that finite-étale towers over $\overline{\mathbb{F}}_q$ are much more flexible than their characteristic-zero analogues.

9. Combining ramification and Prym data

The preceding sections suggest that neither bare ramification nor raw p-rank is likely to be sufficient. The more robust obstruction should combine them.

Let $h : U \to Y$ be a finite separable morphism with $U \to X$ finite étale. The trace-zero sheaf

$$\mathcal{F}_h = h_* \mathbb{Q}_\ell / \mathbb{Q}_\ell$$

on Y has conductor governed by the ramification of h. In the tame case, the conductor is exactly the ramification contribution appearing in Riemann-Hurwitz; in general one must include Swan terms. Thus a sequence with

$$\frac{\deg \mathrm{Diff}_{U/Y}}{\kappa(U)} \to 0$$

should be interpreted as a sequence of high-rank permutation local systems on Y with sublinear conductor, subject to the condition that the source curves U lie in the finite-étale tower of X.

The desired negative theorem would say that small conductor forces the Hasse-Witt/Prym profile of U to resemble that of Y, while the finite-étale tower of a carefully chosen X has an incompatible profile.

Conjecture 9.1 (Ramification–Prym rigidity)**.** *Let Y/k be fixed. There exists a functorial positive-characteristic profile $\mathcal{P}$, built from cyclic Prym p-ranks or generalized Hasse-Witt invariants, such that whenever $h_n : U_n \to Y$ is a sequence of finite separable morphisms with*

$$\frac{\deg \mathrm{Diff}_{U_n/Y}}{\kappa(U_n)} \to 0,$$

the profiles $\mathcal{P}(U_n)$ converge, in an appropriate tower sense, to the profile forced by Y.

Problem 9.2. *Find curves $X, Y/k$ such that the finite-étale tower profile of X is incompatible with the small-ramification profile forced by Y. Such a pair would satisfy $\sigma(X, Y) < 1$ and would answer Question 1.1 negatively.*

10. Summary of proved reductions

The note establishes the following unconditional or explicitly conditional reductions.

(i) The ramification score $\sigma(X, Y)$ is invariant under replacing either variable by a finite étale cover and satisfies a triangle inequality; see Propositions 3.2 and 3.3.

(ii) If $\sigma(X, Y) < 1$ for some pair, then X and Y have no common finite étale cover; see Corollary 3.4.

(iii) If $\sigma(X, Y) = 1$ is attained, then X and Y have a common finite étale cover; see Corollary 3.5.

(iv) High-degree covers of a fixed target with bounded total ramification exist; see Proposition 4.1. Thus bare ramification alone is unlikely to be enough unless the source tower is used.

(v) Assuming the standard Tamagawa-Yang p-average consequence, the lower tail Hasse-Witt defect satisfies $\underline{D}_p(X) = 0$ for every X; see Proposition 5.2. Hence normalized p-rank by itself is too coarse.

(vi) The cyclic Prym defect invariant $M_\ell(X)$ is a finite-étale commensurability invariant; see Proposition 6.3. Its dynamics are governed by the exact formula

$$D_p(V) = \frac{1}{\ell} D_p(U) + \frac{\ell - 1}{\ell} \delta_p(P_{V/U}).$$

(vii) A single strict inequality $M_\ell(X) \neq M_\ell(Y)$ would answer the common-cover question negatively for that pair.

11. Open tasks

The most concrete tasks are the following.

Task 1. Prove or disprove the Prym gap conjecture for $\ell = 2$ and $p \neq 2$: does there exist X/k such that every unramified double-cover Prym in the finite-étale tower of X has p-rank bounded below by a positive proportion of its dimension?

Task 2. Prove or disprove universal Prym-density: does every finite-étale tower contain, arbitrarily far up, cyclic unramified covers with Pryms of arbitrarily small normalized p-rank?

Task 3. Strengthen σ by decorating low-ramification morphisms $U \to Y$ with generalized Hasse-Witt data of cyclic covers of U.

Task 4. Investigate whether virtual tame orbifold uniformization holds over $\overline{\mathbb{F}}_q$. A positive answer would imply that all curves of genus at least 2 have a common finite étale cover.

References

[1] F. Bogomolov and Y. Tschinkel, *Unramified correspondences*, arXiv:math/0202223.

[2] T. O. Celik, Y. Elias, B. Gunes, R. Newton, E. Ozman, R. Pries, and L. Thomas, *Non-ordinary curves with a Prym variety of low p-rank*, in Women in Numbers Europe II, Association for Women in Mathematics Series, vol. 11, Springer, 2018, pp. 117–158.

[3] S. Mochizuki, *Correspondences on hyperbolic curves*, J. Pure Appl. Algebra **131** (1998), no. 3, 227–244.

[4] S. Nakajima, *On generalized Hasse-Witt invariants of an algebraic curve*, Galois groups and their representations, Adv. Stud. Pure Math. **2**, North-Holland, 1983, pp. 69–88.

[5] E. Ozman and R. Pries, *Ordinary and almost ordinary Prym varieties*, Asian J. Math. **23** (2019), no. 3, 455–478.

[6] M. Raynaud, *Revêtements des courbes en caractéristique $p > 0$ et ordinarité*, Compositio Math. **123** (2000), 73–88.

[7] M. Saïdi and A. Tamagawa, *Variation of fundamental groups of curves in positive characteristic*, J. Algebraic Geom. **24** (2015), no. 3, 573–617.

[8] A. Tamagawa, *Finiteness of isomorphism classes of curves in positive characteristic with prescribed fundamental groups*, J. Algebraic Geom. **13** (2004), no. 4, 675–724.

[9] A. Tamagawa, *Correspondences on curves in positive characteristic*, in Abelian Varieties and Number Theory, Contemp. Math. **767**, Amer. Math. Soc., 2021, pp. 97–114.

[10] Y. Yang, *On the averages of generalized Hasse-Witt invariants of pointed stable curves in positive characteristic*, Math. Z. **295** (2020), 1–45.

[11] Y. Yang, *Raynaud-Tamagawa theta divisors and new-ordinariness of ramified coverings of curves*, J. Algebra **587** (2021), 147–172.