COHOMOLOGICAL RIGIDITY FOR MAPPING CLASS GROUP REPRESENTATIONS WITH FINITE TORELLI DATA

Abstract. Let G be a genus $g \geq 3$ mapping class group and let $\rho : G \to \mathrm{GL}(V)$ be a finite-dimensional complex representation. Litt conjectured that irreducible such representations are cohomologically rigid, i.e. that $H^1(G, \operatorname{End} V) = 0$. We prove several unconditional cases. We first record the rigidity of all irreducible representations factoring through the homological quotient $\mathrm{Sp}_{2g}(\mathbb{Z})$, and of all irreducible representations with scalar Torelli action. For finite-image representations, Putman's and Ershov–He's finite-index abelianization theorems imply rigidity under Johnson-kernel or lower-central-series control on the kernel. The main new extension in this note is a stable-range tensor theorem: if $G = \mathrm{Mod}(\Sigma_{g,\epsilon})$, $\epsilon \in \{0, 1\}$, $r \geq 2$, $g \geq 8r - 4$, α is a finite-image representation whose Torelli restriction kills $\gamma_r \mathcal{I}_g^\epsilon$, and B is an algebraic Sp_{2g}-module on which $-I$ acts by a scalar, then

$$H^1\big(G, \mathrm{End}(\alpha \otimes B)\big) = 0.$$

Consequently every irreducible constituent of $\alpha \otimes B$ is rigid. This gives infinite-image examples with genuinely nontrivial finite Torelli monodromy, such as algebraic symplectic twists of the finite actions on roots of the canonical bundle in genus at least twelve. We also record the resulting low-dimensional rigidity consequence of the classification theorem of Kaufmann–Salter–Zhang–Zhong.

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Date: May 23, 2026.

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1. Introduction

Let $\Sigma_{g,b}^p$ denote a connected oriented surface of genus g, with b boundary components and p punctures. Boundary components and punctures are fixed pointwise unless explicitly stated otherwise. Put

$$G_{g,b}^p = \mathrm{PMod}(\Sigma_{g,b}^p).$$

For $g \geq 1$, the action on the first homology of the closed surface obtained by filling punctures and boundary components gives a surjection

$$G_{g,b}^p \twoheadrightarrow \mathrm{Sp}_{2g}(\mathbb{Z}).$$

Its kernel is the Torelli group, denoted $\mathcal{I}_{g,b}^p$. The Johnson kernel $\mathcal{K}_{g,b}^p$ is the subgroup generated by Dehn twists about separating curves.

For a finite-dimensional complex representation

$$\rho : G \longrightarrow \mathrm{GL}(V),$$

we let $\mathrm{End}(V)$ be the associated adjoint module, with G acting by conjugation. We say that ρ is *cohomologically rigid* if

$$H^1(G, \mathrm{End}(V)) = 0.$$

When ρ is irreducible, this is the usual vanishing of infinitesimal deformations modulo infinitesimal conjugation.

The motivating conjecture is due to Litt.

Conjecture 1.1 (Litt)**.** *Let $g \geq 3$. Every irreducible finite-dimensional complex representation of $\mathrm{Mod}_{g,n}$ is cohomologically rigid.*

This conjecture is inspired by the analogy between mapping class groups and higher-rank arithmetic lattices. For lattices, first-cohomology vanishing and superrigidity provide strong constraints on finite-dimensional representations. In mapping class groups the analogous statement is much more mysterious. A positive answer to theorem 1.1 would imply, by a standard induction argument, Ivanov's conjecture that finite-index subgroups of $\mathrm{Mod}_{g,n}$ have finite abelianization for $g \geq 3$. Thus even the finite-image part of the conjecture is tied to a major open problem.

The algebro-geometric motivation is that $\mathrm{Mod}_{g,n}$ is the orbifold fundamental group of $\mathcal{M}_{g,n}$, so its representations are local systems on the moduli stack of curves. General expectations around rigid local systems suggest that rigid local systems should be of geometric origin. See Simpson's construction problem [17], Langer–Simpson [10], and Litt's survey [11].

The purpose of this paper is not to solve theorem 1.1. Rather, we isolate unconditional regions where it holds and enlarge the basic finite-image results

to a stable infinite-image family obtained by tensoring finite Torelli data with algebraic symplectic representations.

Main results. The first result is a useful baseline. It is essentially folklore in the algebraic symplectic case and follows formally from Johnson–Hain theory and Starkov's first-cohomology vanishing theorem for higher-rank lattices.

Theorem 1.2 (Symplectic-quotient representations)**.** *Let $g \geq 3$, and let*

$$\rho : G_{g,b}^p \longrightarrow \mathrm{GL}(V)$$

be an irreducible finite-dimensional complex representation which factors through

$$G_{g,b}^p \twoheadrightarrow \mathrm{Sp}_{2g}(\mathbb{Z}).$$

Then

$$H^1(G_{g,b}^p, \mathrm{End}(V)) = 0.$$

The same argument proves the following extension.

Theorem 1.3 (Scalar Torelli action)**.** *Let $g \geq 3$, let $G = G_{g,b}^p$, and let $\mathcal{I} = \mathcal{I}_{g,b}^p$. Let*

$$\rho : G \longrightarrow \mathrm{GL}(V)$$

be irreducible. If

$$\rho(\mathcal{I}) \subseteq \mathbb{C}^\times \cdot I_V,$$

then

$$H^1(G, \mathrm{End}(V)) = 0.$$

For finite-image representations, rigidity follows from finite-index abelianization theorems. The first such result uses Putman's theorem for subgroups containing a large piece of the Johnson kernel.

Theorem 1.4 (Finite image and Johnson-kernel control)**.** *Let $g \geq 3$, and let*

$$\rho : G_{g,b}^p \longrightarrow \mathrm{GL}(V)$$

be a finite-image complex representation. Let $\Lambda = \ker(\rho)$. Suppose that Λ contains a large piece of the Johnson kernel in Putman's sense: if $\mathcal{K}(\Lambda)$ denotes the subgroup of $\Lambda \cap \mathcal{K}_{g,b}^p$ generated by all powers $T_\gamma^m \in \Lambda$ of Dehn twists about separating curves, then

$$[\Lambda \cap \mathcal{K}_{g,b}^p : \mathcal{K}(\Lambda)] < \infty.$$

Then

$$H^1(G_{g,b}^p, \mathrm{End}(V)) = 0.$$

In particular, the same conclusion holds if $\mathcal{K}_{g,b}^p \subseteq \ker(\rho)$.

The second finite-image result uses Ershov–He's theorem. Here $\gamma_r L$ denotes the r-th term of the lower central series of a group L, with $\gamma_1 L = L$ and $\gamma_{r+1} L = [L, \gamma_r L]$.

Theorem 1.5 (Finite image with nilpotent Torelli image)**.** *Let $r \geq 2$, let $g \geq 8r-4$, let $\epsilon \in \{0,1\}$, and put*

$$G = \mathrm{Mod}(\Sigma_{g,\epsilon}), \qquad \mathcal{I} = \mathcal{I}_g^{\epsilon}.$$

If

$$\rho : G \longrightarrow \mathrm{GL}(V)$$

is a finite-image complex representation such that

$$\gamma_r \mathcal{I} \subseteq \ker(\rho),$$

then

$$H^1(G, \mathrm{End}(V)) = 0.$$

Equivalently, in this stable range, every finite-image representation whose Torelli image is nilpotent of class $< r$ is cohomologically rigid.

The bound $g \geq 8r-4$ is the bound in the published version of Ershov–He; the older preprint bound $g \geq 12(r-1)$ is not optimal.

The main additional result of this note is that, in the same Ershov–He stable range, one can tensor such finite Torelli data with algebraic symplectic representations and retain rigidity. This produces infinite-image examples not factoring through $\mathrm{Sp}_{2g}(\mathbb{Z})$ whenever the finite factor sees Torelli.

Theorem 1.6 (Algebraic symplectic twists of finite Torelli data)**.** *Let $r \geq 2$, let $g \geq 8r-4$, let $\epsilon \in \{0,1\}$, and put*

$$G = \mathrm{Mod}(\Sigma_{g,\epsilon}), \qquad \mathcal{I} = \mathcal{I}_g^{\epsilon}.$$

Let

$$\alpha : G \longrightarrow \mathrm{GL}(A)$$

be a finite-image complex representation satisfying

$$\gamma_r \mathcal{I} \subseteq \ker(\alpha|_{\mathcal{I}}).$$

Let B be a finite-dimensional algebraic representation of $\mathrm{Sp}_{2g,\mathbb{C}}$, viewed as a representation of G through the homological quotient, and assume that the central element $-I \in \mathrm{Sp}_{2g}$ acts by a scalar on B. Then

$$H^1\big(G, \mathrm{End}(A \otimes B)\big) = 0.$$

Consequently every irreducible direct summand of $A \otimes B$ is cohomologically rigid.

The hypothesis on $-I$ is automatic if B is irreducible as an algebraic Sp_{2g}-module. It is also automatic for any direct sum of irreducibles of a fixed central parity. It should not simply be omitted: if B has mixed central parity, then $\mathrm{End}(B)$ may contain odd symplectic summands, and the parity argument used below no longer applies.

A concrete family comes from Hain's description of the mapping class group action on roots of the canonical bundle. In genus $g \geq 12$, the Torelli

action in these finite representations factors through Johnson's abelian quotient, so $r = 2$ applies. Tensoring with any irreducible algebraic symplectic representation gives infinite-image representations with finite nontrivial Torelli monodromy; every irreducible constituent is rigid by theorem 1.6.

Finally, a recent low-dimensional classification theorem gives the following independent island of evidence.

Corollary 1.7 (Low-dimensional irreducibles)**.** *Let S be a closed genus $g \geq 3$ surface, or a genus $g \geq 3$ surface with at most one puncture or boundary component. Let*

$$\rho : \mathrm{Mod}(S) \longrightarrow \mathrm{GL}(V)$$

be irreducible and suppose

$$\dim V \leq 3g - 3.$$

Then ρ is cohomologically rigid.

Indeed, Kaufmann–Salter–Zhang–Zhong classify all representations in this range; the irreducible ones are trivial or standard homological constituents, hence are covered by theorem 1.2. We do not assert that every nonsemisimple representation in this dimension range has vanishing adjoint cohomology.

Literature comparison and novelty. The proof of theorem 1.2 uses only standard inputs: Starkov's first-cohomology vanishing theorem for lattices [18] and Johnson–Hain's computation of the rational abelianization of Torelli groups [5, 8]. Litt explicitly records rigidity for irreducible subobjects of tensor powers of the standard symplectic representation [11]. The formulation for arbitrary irreducible finite-dimensional representations of $\mathrm{Sp}_{2g}(\mathbb{Z})$ is a convenient standalone consequence of Starkov's theorem.

The finite-image results theorems 1.4 and 1.5 are formal consequences of known finite-index abelianization theorems: Putman's theorem for finite-index subgroups containing large Johnson-kernel pieces [15], and Ershov–He's theorem for finite-index subgroups containing lower-central-series terms of Torelli groups [3]. These results give rigidity for non-homological finite quotients, including finite quotients that remember Johnson data.

The tensor theorem theorem 1.6 is the main new packaging in this note. It uses the second, Torelli-internal part of Ershov–He's theorem, namely that finite-index subgroups of $\mathcal{I}$ containing $\gamma_r\mathcal{I}$ have the same rational first cohomology as $\mathcal{I}$, together with a Borel-density parity lemma. This allows finite nilpotent Torelli data to be combined with arbitrary algebraic symplectic monodromy of pure central parity.

Other important evidence for Litt's conjecture comes from methods not used here. Looijenga initiated the study of Prym representations [12]; Putman–Wieland related abelian quotients of finite-index subgroups to higher Prym representations [16]; Zhong proves infinitesimal rigidity for symplectic Prym representations associated to finite abelian regular covers of non-closed finite-type surfaces [19]. Godfard proves rigidity of Witten–Reshetikhin–Turaev

SU(2) and SO(3) quantum representations at prime levels for closed surfaces of genus at least seven [4]. These families have genuinely non-scalar Torelli behavior and are not subsumed by the elementary arguments below.

2. Preliminaries

2.1. **Cohomology and deformation theory.** Let G be a discrete group and M a complex G-module. We write $H^\bullet(G, M)$ for group cohomology. If $\rho : G \to \mathrm{GL}(V)$ is a finite-dimensional complex representation, then $\mathrm{End}(V)$ is a G-module by

$$g \cdot A = \rho(g) A \rho(g)^{-1}.$$

The vector space $Z^1(G, \mathrm{End}(V))$ is the Zariski tangent space at ρ to $\mathrm{Hom}(G, \mathrm{GL}(V))$, and $B^1(G, \mathrm{End}(V))$ is the tangent space to the conjugation orbit. Thus $H^1(G, \mathrm{End}(V)) = 0$ implies infinitesimal rigidity modulo conjugation.

We use repeatedly the following standard facts.

Lemma 2.1 (Finite-index restriction)**.** *Let $H \leq G$ be a finite-index subgroup and let M be a complex G-module. Then the restriction map*

$$\mathrm{res} : H^1(G, M) \longrightarrow H^1(H, M)$$

is injective.

Proof. The corestriction–restriction composite is multiplication by $[G : H]$. Since the coefficients have characteristic zero, restriction is injective. □

Lemma 2.2 (Inflation-restriction)**.** *Let*

$$1 \longrightarrow N \longrightarrow G \longrightarrow Q \longrightarrow 1$$

be a short exact sequence, and let M be a G-module. Then there is an exact sequence

$$0 \longrightarrow H^1(Q, M^N) \longrightarrow H^1(G, M) \longrightarrow H^1(N, M)^Q.$$

If N acts trivially on M, then

$$H^1(N, M) \cong \mathrm{Hom}(H_1(N, \mathbb{C}), M).$$

Proof. This is the low-degree Hochschild–Serre exact sequence. If N acts trivially on M, then crossed homomorphisms $N \to M$ are ordinary homomorphisms and coboundaries vanish. □

2.2. **Mapping class group notation.** Let

$$H_{\mathbb{Z}} = H_1(\Sigma_g, \mathbb{Z}), \qquad H = H_{\mathbb{Z}} \otimes \mathbb{C}.$$

For $g \geq 1$, the homological representation gives a surjection

$$G^p_{g,b} \twoheadrightarrow \mathrm{Sp}(H_{\mathbb{Z}}) \cong \mathrm{Sp}_{2g}(\mathbb{Z}).$$

We write

$$\mathcal{I}^p_{g,b} = \ker\{G^p_{g,b} \to \mathrm{Sp}_{2g}(\mathbb{Z})\}.$$

When the surface is closed, we write $G_g = \mathrm{Mod}(\Sigma_g)$, $\mathcal{I}_g = \mathcal{I}^0_{g,0}$, and $\mathcal{K}_g = \mathcal{K}^0_{g,0}$. When the surface has one boundary component, we write $G^1_g = \mathrm{Mod}(\Sigma_{g,1})$ and $\mathcal{I}^1_g = \mathcal{I}^0_{g,1}$.

The following computation is the key input from Johnson theory.

Theorem 2.3 (Johnson–Hain–Putman)**.** *For $g \geq 3$, there is an $\mathrm{Sp}_{2g}(\mathbb{Z})$-equivariant isomorphism*

$$H_1(\mathcal{I}^p_{g,b}, \mathbb{C}) \cong (\Lambda^3 H/H) \oplus H^{\oplus(b+p)}.$$

Equivalently, as an algebraic Sp_{2g}-module,

$$H_1(\mathcal{I}^p_{g,b}, \mathbb{C}) \cong V(\lambda_3) \oplus V(\lambda_1)^{\oplus(b+p)}.$$

In particular, the central element $-I \in \mathrm{Sp}_{2g}(\mathbb{Z})$ acts by -1 on $H_1(\mathcal{I}^p_{g,b}, \mathbb{C})$.

Proof. Johnson computed the abelianization of the Torelli group in the closed and one-boundary cases, and Hain records the punctured form. Putman states the bordered and punctured quotient in the form

$$\mathcal{I}^p_{g,b}/\mathcal{K}^p_{g,b} \cong (\Lambda^3 H_{\mathbb{Z}}/H_{\mathbb{Z}}) \oplus H_{\mathbb{Z}}^{\oplus(b+p)}$$

up to torsion, and this identifies the rational abelianization; see [5, Section 3] and [15, Lemma 4.3]. Tensoring with $\mathbb{C}$ gives the displayed formula. Since $-I$ acts by -1 on H, it acts by -1 on both H and $\Lambda^3 H/H$. □

We also use perfectness of pure mapping class groups.

Theorem 2.4 (Powell; finite-type pure form)**.** *If $g \geq 3$, then the finite-type pure mapping class group $G^p_{g,b}$ is perfect:*

$$H_1(G^p_{g,b}, \mathbb{Z}) = 0.$$

Proof. For closed surfaces this is Powell's theorem [14]. The finite-type pure statement with punctures and boundary is standard and is recalled, for example, in [1]. □

2.3. **Lattice-cohomology and density inputs.** The analytic input is Starkov's theorem.

Theorem 2.5 (Starkov)**.** *Let L be a connected semisimple Lie group and $\Gamma \leq L$ a lattice. Assume that there is no epimorphism from L onto a Lie group locally isomorphic to $\mathrm{SO}(1,n)$ or $\mathrm{SU}(1,n)$ whose image of Γ is a lattice. Then*

$$H^1(\Gamma, W) = 0$$

for every finite-dimensional real representation W of Γ.

For $L = \mathrm{Sp}_{2g}(\mathbb{R})$, $g \geq 2$, the hypothesis is satisfied. We apply this to realifications of complex representations.

We will also use the following elementary Borel-density consequence.

Lemma 2.6 (Parity with a finite-image factor)**.** *Let $\Gamma \leq \mathrm{Sp}_{2g}(\mathbb{Z})$ be a finite-index subgroup. Let W and U be finite-dimensional algebraic $\mathrm{Sp}_{2g,\mathbb{C}}$-modules. Assume that all irreducible constituents of W are odd under the central element $-I$, and all irreducible constituents of U are even under $-I$. Let C be a finite-image complex Γ-module. Then*

$$\mathrm{Hom}_\Gamma(W, C \otimes U) = 0.$$

Proof. Let $\Gamma_0 \leq \Gamma$ be a finite-index subgroup acting trivially on C. If a Γ-equivariant map $W \to C \otimes U$ existed, then after restriction to Γ_0 it would give a nonzero element of $\mathrm{Hom}_{\Gamma_0}(W, U^{\oplus \dim C})$. Since Γ_0 is Zariski dense in $\mathrm{Sp}_{2g,\mathbb{C}}$, such a map is $\mathrm{Sp}_{2g,\mathbb{C}}$-equivariant. But an Sp_{2g}-equivariant map from an odd central-parity module to an even central-parity module is zero. □

2.4. **Ershov–He's theorem.** We need two consequences of Ershov–He. We state them in the mapping-class-group case.

Theorem 2.7 (Ershov–He)**.** *Let $r \geq 2$, let $g \geq 8r - 4$, let $\epsilon \in \{0, 1\}$, and put*

$$G = \mathrm{Mod}(\Sigma_{g,\epsilon}), \qquad \mathcal{I} = \mathcal{I}_g^\epsilon.$$

Then:

1. *If $\Lambda \leq G$ is finite-index and $\gamma_r \mathcal{I} \subseteq \Lambda$, then Λ^{ab} is finite.*
2. *If $L \leq \mathcal{I}$ is finite-index and $\gamma_r \mathcal{I} \subseteq L$, then the restriction map*

$$H^1(\mathcal{I}, \mathbb{C}) \longrightarrow H^1(L, \mathbb{C})$$

is an isomorphism.

Proof. This is the mapping-class-group part of [3, Theorem 1.7], using the stable range from the published version. Part (1) is the finite-abelianization statement for finite-index subgroups of the ambient mapping class group. Part (2) is the corresponding Torelli-internal restriction statement. □

3. The symplectic quotient and scalar Torelli action

We first prove theorem 1.2; then we prove the scalar Torelli variant.

Theorem 3.1 (Symplectic quotient rigidity)**.** *Let $g \geq 3$. Let*

$$\rho : G_{g,b}^p \longrightarrow \mathrm{GL}(V)$$

be an irreducible finite-dimensional complex representation which factors through $\mathrm{Sp}_{2g}(\mathbb{Z})$. Then

$$H^1(G_{g,b}^p, \mathrm{End}(V)) = 0.$$

Proof. Put $G = G_{g,b}^p$, $\mathcal{I} = \mathcal{I}_{g,b}^p$, $\Gamma = \mathrm{Sp}_{2g}(\mathbb{Z})$, and $M = \mathrm{End}(V)$. Since ρ factors through Γ, the group $\mathcal{I}$ acts trivially on M. Hochschild–Serre for

$$1 \longrightarrow \mathcal{I} \longrightarrow G \longrightarrow \Gamma \longrightarrow 1$$

gives

$$0 \longrightarrow H^1(\Gamma, M) \longrightarrow H^1(G, M) \longrightarrow H^1(\mathcal{I}, M)^\Gamma.$$

By Starkov's theorem, $H^1(\Gamma, M) = 0$. Since $\mathcal{I}$ acts trivially on M,

$$H^1(\mathcal{I}, M)^\Gamma \cong \mathrm{Hom}_\Gamma(H_1(\mathcal{I}, \mathbb{C}), M).$$

By theorem 2.3, $-I \in \Gamma$ acts by -1 on $H_1(\mathcal{I}, \mathbb{C})$. Since $-I$ is central in Γ and ρ is irreducible, Schur's lemma implies that $\rho(-I)$ is scalar. Thus $-I$ acts trivially on $M = \mathrm{End}(V)$. Therefore

$$\mathrm{Hom}_\Gamma(H_1(\mathcal{I}, \mathbb{C}), M) = 0,$$

and the theorem follows. □

Remark 3.2 (Geometric origin in the algebraic symplectic sector). If V is an irreducible algebraic Sp_{2g}-module, then the corresponding local system is visibly geometric. The standard symplectic local system H is $R^1 f_* \mathbb{C}$ for the universal curve, and every algebraic irreducible Sp_{2g}-module occurs as a subquotient of tensor powers of H. By Kunneth, these tensor powers occur in cohomology of fiber powers of the universal curve.

The same argument gives a useful reduction for arbitrary representations.

Proposition 3.3 (Reduction to Torelli cohomology)**.** *Let $g \geq 3$, let $G = G^p_{g,b}$, let $\mathcal{I} = \mathcal{I}^p_{g,b}$, and let $\Gamma = \mathrm{Sp}_{2g}(\mathbb{Z})$. For every finite-dimensional complex representation $\rho : G \to \mathrm{GL}(V)$, with $M = \mathrm{End}(V)$, there is an injection*

$$H^1(G, M) \hookrightarrow H^1(\mathcal{I}, M)^\Gamma.$$

Proof. The Hochschild–Serre sequence gives

$$0 \to H^1(\Gamma, M^{\mathcal{I}}) \to H^1(G, M) \to H^1(\mathcal{I}, M)^\Gamma.$$

The space $M^{\mathcal{I}}$ is a finite-dimensional Γ-module, so Starkov's theorem gives $H^1(\Gamma, M^{\mathcal{I}}) = 0$. □

Theorem 3.4 (Scalar Torelli action)**.** *Let $g \geq 3$, let $G = G^p_{g,b}$, and let $\mathcal{I} = \mathcal{I}^p_{g,b}$. Let*

$$\rho : G \longrightarrow \mathrm{GL}(V)$$

be irreducible. If $\rho(\mathcal{I}) \subseteq \mathbb{C}^\times \cdot I_V$, then

$$H^1(G, \mathrm{End}(V)) = 0.$$

Proof. Put $M = \mathrm{End}(V)$. Since $\mathcal{I}$ acts by scalars on V, it acts trivially on M. As above, Hochschild–Serre and Starkov reduce the problem to proving

$$\mathrm{Hom}_\Gamma(H_1(\mathcal{I}, \mathbb{C}), M) = 0.$$

Choose a lift $s \in G$ of $-I \in \Gamma$. Since $-I$ is central in Γ, for every $a \in G$ one has $[s, a] \in \mathcal{I}$. By the scalar Torelli hypothesis,

$$\rho(s)\rho(a)\rho(s)^{-1} = c(a)\rho(a)$$

for some $c(a) \in \mathbb{C}^\times$. The function $c : G \to \mathbb{C}^\times$ is a character. By theorem 2.4, G is perfect, so $c = 1$. Hence $\rho(s)$ commutes with $\rho(G)$. Since ρ is irreducible, Schur's lemma implies that $\rho(s)$ is scalar. Therefore $-I$ acts trivially on M. Since $-I$ acts by -1 on $H_1(\mathcal{I}, \mathbb{C})$, no nonzero equivariant map can exist. □

Corollary 3.5 (Unlabelled punctures and other finite extensions)**.** *Let $\widehat{G}$ be a finite extension of one of the pure groups $G^p_{g,b}$, for instance the mapping class group in which punctures may be permuted. If a representation of $\widehat{G}$ restricts to one of the rigid representations covered by theorems 3.1 and 3.4 on $G^p_{g,b}$, then it is rigid as a representation of $\widehat{G}$.*

Proof. Apply theorem 2.1 to the finite-index subgroup $G^p_{g,b} \leq \widehat{G}$. □

4. Finite-image representations

A finite-image representation is rigid whenever the abelianization of its kernel is rationally trivial.

Proposition 4.1 (Finite-image FAb criterion)**.** *Let G be a group and let*

$$\rho : G \longrightarrow \mathrm{GL}(V)$$

be a finite-image complex representation. Put $M = \mathrm{End}(V)$, $Q = \rho(G)$, and $L = \ker(\rho)$. Then there is a natural isomorphism

$$H^1(G, M) \cong \mathrm{Hom}_Q(H_1(L, \mathbb{C}), M).$$

In particular, if $H_1(L, \mathbb{C}) = 0$, then $H^1(G, M) = 0$.

Proof. The group L acts trivially on M, and we have an exact sequence

$$1 \longrightarrow L \longrightarrow G \longrightarrow Q \longrightarrow 1.$$

Since Q is finite and the coefficients are complex, $H^i(Q, W) = 0$ for every complex Q-module W and every $i > 0$. Hochschild–Serre gives

$$H^1(G, M) \cong H^1(L, M)^Q.$$

Since L acts trivially on M,

$$H^1(L, M) \cong \mathrm{Hom}(H_1(L, \mathbb{C}), M).$$

Taking Q-invariants gives the result. □

4.1. Johnson-kernel control.

For a finite-index subgroup $\Lambda \leq G^p_{g,b}$, define $\mathcal{K}(\Lambda)$ to be the subgroup of $\Lambda \cap \mathcal{K}^p_{g,b}$ generated by all elements $T^m_\gamma \in \Lambda$, where γ ranges over separating curves and $m \in \mathbb{Z}$.

Theorem 4.2 (Putman)**.** *Let $g \geq 3$, and let $\Lambda \leq G^p_{g,b}$ be a finite-index subgroup. If*

$$[\Lambda \cap \mathcal{K}^p_{g,b} : \mathcal{K}(\Lambda)] < \infty,$$

then

$$H_1(\Lambda, \mathbb{R}) = 0.$$

In particular, this holds if $\mathcal{K}^p_{g,b} \subseteq \Lambda$.

Proof. This is [15, Theorem B and Corollary C]. □

Theorem 4.3 (Rigidity from Johnson-kernel control)**.** *Let $g \geq 3$, and let*

$$\rho : G_{g,b}^p \longrightarrow \mathrm{GL}(V)$$

be a finite-image complex representation. Set $\Lambda = \ker(\rho)$. If

$$[\Lambda \cap \mathcal{K}_{g,b}^p : \mathcal{K}(\Lambda)] < \infty,$$

then

$$H^1(G_{g,b}^p, \mathrm{End}(V)) = 0.$$

In particular, the conclusion holds if $\mathcal{K}_{g,b}^p \subseteq \ker(\rho)$.

Proof. By theorem 4.2, $H_1(\Lambda, \mathbb{R}) = 0$, hence $H_1(\Lambda, \mathbb{C}) = 0$. Apply theorem 4.1. □

Example 4.4 (Finite Johnson congruence quotients). Let $A_{g,b}^p = \mathcal{I}_{g,b}^p / \mathcal{K}_{g,b}^p$. By theorem 2.3, this is a finite-rank abelian group up to torsion, with rationalization

$$(\Lambda^3 H / H) \oplus H^{\oplus(b+p)}.$$

For any $m \geq 1$, one obtains finite quotients of $G_{g,b}^p$ by reducing the Johnson quotient modulo m and combining this with a congruence quotient of $\mathrm{Sp}_{2g}(\mathbb{Z})$. Any finite-dimensional representation of such a finite quotient is rigid by theorem 4.3, and many of these representations have nontrivial Torelli action.

Example 4.5 (Roots of the canonical bundle). Hain describes the action of Γ_g on the finite set Θ_n of n-th roots of the canonical bundle of a curve. The full automorphism group of this affine $H_1(C, \mathbb{Z}/n\mathbb{Z})$-space fits into an exact sequence

$$0 \longrightarrow H_1(C, \mathbb{Z}/n\mathbb{Z}) \longrightarrow \mathrm{Aut}(\Theta_n) \longrightarrow \mathrm{GL}(H_1(C, \mathbb{Z}/n\mathbb{Z})) \longrightarrow 1,$$

and the mapping class group image lies over the symplectic subgroup. Hain computes the restriction to the Torelli group using Johnson's homomorphism; in particular, the Torelli action is trivial if and only if n divides 2, and otherwise it is nontrivial but factors through the abelian Johnson quotient [5, Theorem 13.3]. Therefore irreducible constituents of the corresponding permutation representations are finite-image representations with nontrivial Torelli action and are rigid by theorem 4.3.

4.2. **Lower central series of Torelli groups.** Combining theorem 4.1 with theorem 2.7(1) gives the stable nilpotent-Torelli finite-image result.

Theorem 4.6 (Rigidity with nilpotent Torelli image)**.** *Let $r \geq 2$, let $g \geq 8r - 4$, let $\epsilon \in \{0, 1\}$, and put*

$$G = \mathrm{Mod}(\Sigma_{g,\epsilon}), \qquad \mathcal{I} = \mathcal{I}_g^\epsilon.$$

If

$$\rho : G \longrightarrow \mathrm{GL}(V)$$

is a finite-image complex representation satisfying

$$\gamma_r\mathcal{I} \subseteq \ker(\rho),$$

then

$$H^1(G, \mathrm{End}(V)) = 0.$$

Proof. Set $\Lambda = \ker(\rho)$. By theorem 2.7(1), $H_1(\Lambda, \mathbb{C}) = 0$. The result follows from theorem 4.1. □

Corollary 4.7 (Abelian Torelli image)**.** *For $g \geq 12$, every finite-image complex representation of $\mathrm{Mod}(\Sigma_{g,1})$ or $\mathrm{Mod}(\Sigma_g)$ whose Torelli image is abelian is cohomologically rigid.*

Proof. This is the case $r = 2$, since $\gamma_2\mathcal{I} = [\mathcal{I}, \mathcal{I}]$. □

Corollary 4.8 (Projectively finite image)**.** *Let $g \geq 3$, and let G be either a pure finite-type genus g mapping class group or a finite extension of one. If $\rho : G \to \mathrm{GL}(V)$ has finite projective image, then $\rho(G)$ is finite. Consequently the finite-image criteria above apply to projectively finite representations.*

Proof. The groups under consideration have finite abelianization; in the pure case they are perfect by theorem 2.4, and finite extensions have finite abelianization. Hence $\det \rho(G)$ is finite. If the projective image is finite, then the scalar subgroup of $\rho(G)$ has finite determinant image; in dimension d, a scalar λI in $\rho(G)$ satisfies $\lambda^d \in \det \rho(G)$, so only finitely many such scalars occur. Thus $\rho(G)$ is finite. □

5. Algebraic symplectic twists of finite Torelli data

This section proves theorem 1.6. The point is that the kernel of the finite Torelli action is not a finite-index subgroup of the whole mapping class group, so the finite-image FAb criterion cannot be applied directly. Instead, one uses the Torelli-internal part of Ershov–He's theorem to identify its rational first homology with $H_1(\mathcal{I}, \mathbb{C})$, and then uses central parity.

Theorem 5.1 (Tensor rigidity in the Ershov–He range)**.** *Let $r \geq 2$, let $g \geq 8r - 4$, let $\epsilon \in \{0, 1\}$, and put*

$$G = \mathrm{Mod}(\Sigma_{g,\epsilon}), \qquad \mathcal{I} = \mathcal{I}_g^\epsilon, \qquad \Gamma = \mathrm{Sp}_{2g}(\mathbb{Z}).$$

Let

$$\alpha : G \longrightarrow \mathrm{GL}(A)$$

be a finite-image complex representation such that

$$\gamma_r\mathcal{I} \subseteq \ker(\alpha|_{\mathcal{I}}).$$

Let B be a finite-dimensional algebraic $\mathrm{Sp}_{2g,\mathbb{C}}$-module, viewed as a G-module through $G \to \Gamma$, and assume that $-I \in \mathrm{Sp}_{2g}$ acts by a scalar on B. Then, for

$$V = A \otimes B,$$

one has

$$H^1(G, \mathrm{End}(V)) = 0.$$

Proof. Let

$$N = \ker(\alpha|_{\mathcal{I}}).$$

Then $N \triangleleft G$, $N \leq \mathcal{I}$, and $\mathcal{I}/N$ is finite. The group N acts trivially on A, and it acts trivially on B because B factors through $G/\mathcal{I}$. Thus N acts trivially on

$$M = \mathrm{End}(V) = \mathrm{End}(A) \otimes \mathrm{End}(B).$$

Let $E = G/N$ and $F = \mathcal{I}/N$. There is an exact sequence

$$1 \longrightarrow F \longrightarrow E \longrightarrow \Gamma \longrightarrow 1$$

with F finite. Hochschild–Serre for this sequence gives

$$0 \to H^1(\Gamma, M^F) \to H^1(E, M) \to H^1(F, M)^\Gamma.$$

The right term vanishes because F is finite and the coefficients are complex. The left term vanishes by Starkov's theorem. Hence

$$H^1(E, M) = 0.$$

Now apply Hochschild–Serre to

$$1 \longrightarrow N \longrightarrow G \longrightarrow E \longrightarrow 1.$$

Since N acts trivially on M, we get an injection

$$H^1(G, M) \hookrightarrow \mathrm{Hom}(H_1(N, \mathbb{C}), M)^E.$$

By theorem 2.7(2), the inclusion $N \hookrightarrow \mathcal{I}$ induces an isomorphism

$$H_1(N, \mathbb{C}) \xrightarrow{\sim} H_1(\mathcal{I}, \mathbb{C})$$

dual to the restriction isomorphism on $H^1(-, \mathbb{C})$. This isomorphism is E-equivariant. The finite group $F = \mathcal{I}/N$ acts trivially on $H_1(\mathcal{I}, \mathbb{C})$, so

$$\mathrm{Hom}(H_1(N, \mathbb{C}), M)^E \cong \mathrm{Hom}_\Gamma\big(H_1(\mathcal{I}, \mathbb{C}), M^F\big).$$

The F-invariants of M are

$$M^F = (\mathrm{End}\, A)^{\alpha(\mathcal{I})} \otimes \mathrm{End}(B).$$

The first factor $(\mathrm{End}\, A)^{\alpha(\mathcal{I})}$ is a finite-image Γ-module, because $\alpha(G)$ is finite. By theorem 2.3, every irreducible algebraic constituent of $H_1(\mathcal{I}, \mathbb{C})$ is odd under $-I$. Since $-I$ acts by a scalar on B, it acts trivially on $\mathrm{End}(B)$, so every algebraic constituent of $\mathrm{End}(B)$ is even under $-I$. The parity-density lemma theorem 2.6 gives

$$\mathrm{Hom}_\Gamma\big(H_1(\mathcal{I}, \mathbb{C}), M^F\big) = 0.$$

Therefore $H^1(G, M) = 0$. □

Corollary 5.2 (Irreducible constituents)**.** *In the setting of theorem 5.1, every irreducible direct summand of $A \otimes B$ is cohomologically rigid.*

Proof. The representation A has finite image, hence is semisimple over $\mathbb{C}$. The representation B is algebraic for the reductive group $\mathrm{Sp}_{2g,\mathbb{C}}$, hence is semisimple. The Zariski closure of the image on $A \otimes B$ is a finite extension of a reductive group, so $A \otimes B$ is semisimple as a G-module. If

$$A \otimes B \cong \bigoplus_i V_i$$

is a decomposition into irreducible G-modules, then

$$\mathrm{End}(A \otimes B) \cong \bigoplus_{i,j} \mathrm{Hom}(V_i, V_j)$$

as G-modules. In particular, each $\mathrm{End}(V_i)$ is a direct summand of $\mathrm{End}(A \otimes B)$. Group cohomology commutes with finite direct sums, so $H^1(G, \mathrm{End}(V_i))$ is a direct summand of $H^1(G, \mathrm{End}(A \otimes B)) = 0$. □

Example 5.3 (Algebraic twists of canonical-root actions). Let $g \geq 12$, and let α be a finite-dimensional constituent of the permutation representation associated to the action on n-th roots of the canonical bundle. By Hain's computation, $\alpha|_{\mathcal{I}}$ factors through the abelian Johnson quotient, so $\gamma_2\mathcal{I} \subseteq \ker(\alpha|_{\mathcal{I}})$. Let B be any irreducible algebraic Sp_{2g}-module. Then every irreducible constituent of $\alpha \otimes B$ is rigid. If $n \nmid 2$ and B has infinite symplectic monodromy, these constituents typically have both nontrivial finite Torelli monodromy and infinite image.

Example 5.4 (Nilpotent Torelli quotients and algebraic coefficients). Fix $r \geq 2$, $g \geq 8r - 4$, and $\epsilon \in \{0, 1\}$. Let Q be any finite quotient of $\mathrm{Mod}(\Sigma_{g,\epsilon})$ in which $\gamma_r\mathcal{I}_g^\epsilon$ maps trivially on the Torelli subgroup. For every finite-dimensional representation A of Q, and every irreducible algebraic Sp_{2g}-module B, every irreducible constituent of $A \otimes B$ is rigid.

Remark 5.5 (Why the Putman theorem is not enough for the tensor result). It is tempting to replace the Ershov–He lower-central-series hypothesis in theorem 5.1 by the assumption that $\ker(\alpha|_{\mathcal{I}})$ contains the Johnson kernel. The argument above would then require control of $H_1(N, \mathbb{C})$ for finite-index subgroups $N \leq \mathcal{I}$ containing $\mathcal{K}$. Putman's theorem controls the abelianization of finite-index subgroups of the ambient mapping class group containing large pieces of $\mathcal{K}$, not the abelianization of finite-index subgroups of $\mathcal{I}$ itself. Thus the tensor theorem is stated only in the stable Ershov–He range where the required Torelli-internal statement is available.

6. Low-dimensional irreducibles

We record the low-dimensional consequence of the classification theorem of Kaufmann–Salter–Zhang–Zhong.

Theorem 6.1 (Kaufmann–Salter–Zhang–Zhong, paraphrased)**.** *Let S be a closed genus $g \geq 3$ surface, or a genus $g \geq 3$ surface with at most one puncture or boundary component. Every complex representation of* $\mathrm{Mod}(S)$ *of dimension at most $3g-3$ is a direct sum of trivial representations and standard*

homological representations: in the closed case the standard $H_1(\Sigma_g)$ representation, and in the one-puncture or one-boundary case the corresponding homology or unit-tangent-bundle homology representation and its dual.

Proof. This is the main classification theorem of [9]; the formulation here is only the part needed for the corollary below. □

Corollary 6.2 (Low-dimensional rigidity)**.** *Let S be as in theorem 6.1. If*

$$\rho : \mathrm{Mod}(S) \to \mathrm{GL}(V)$$

is irreducible and $\dim V \leq 3g-3$, then $H^1(\mathrm{Mod}(S), \mathrm{End}(V)) = 0$.

Proof. By theorem 6.1, the irreducible constituents in this range are trivial or homological. They factor through $\mathrm{Sp}_{2g}(\mathbb{Z})$, so theorem 3.1 applies. For the full group with a puncture, one can also restrict to the pure subgroup and use theorem 2.1. □

7. Consequences, limitations, and next cases

The results above establish theorem 1.1 in several regions:

(1) irreducible representations factoring through $\mathrm{Sp}_{2g}(\mathbb{Z})$;
(2) irreducible representations with scalar Torelli action;
(3) finite-image representations whose kernels satisfy Putman's Johnson-kernel condition;
(4) finite-image representations whose Torelli image is nilpotent in the Ershov–He stable range;
(5) irreducible constituents of finite nilpotent Torelli data tensored with pure-parity algebraic symplectic representations in the same stable range;
(6) irreducible representations in the low-dimensional range covered by Kaufmann–Salter–Zhang–Zhong.

The tensor theorem gives a supply of infinite-image, non-homological examples. These are still quite controlled: the non-homological part is finite and nilpotent on Torelli, while the infinite part is algebraic symplectic. From the algebro-geometric viewpoint, this is a natural class. Finite monodromy local systems arise as direct summands of pushforwards from finite etale covers of the moduli stack, and algebraic symplectic local systems arise from the cohomology of fiber powers of the universal curve.

The hard frontier remains the case of genuinely non-scalar, infinite Torelli monodromy. By theorem 3.3, for any representation $\rho : G^p_{g,b} \to \mathrm{GL}(V)$, all possible non-rigidity injects into

$$H^1(\mathcal{I}^p_{g,b}, \mathrm{End}(V))^{\mathrm{Sp}_{2g}(\mathbb{Z})}.$$

Thus new progress likely requires computations of twisted Torelli cohomology. This is precisely the terrain of Prym and quantum representations, where existing rigidity results use substantially deeper input.

Question 7.1 (Finite-image frontier)**.** *Let $g \geq 3$, and let $\rho : G^p_{g,b} \to \mathrm{GL}(V)$ have finite image. By theorem 4.1, finite-image rigidity is equivalent to*

$$\mathrm{Hom}_{\rho(G)}(H_1(\ker \rho, \mathbb{C}), \mathrm{End}(V)) = 0.$$

Can this be proved for finite quotients whose kernels do not contain a known Johnson-kernel piece or a known lower-central-series term of Torelli?

Question 7.2 (Torelli twisted cohomology)**.** *For which naturally occurring infinite-image representations ρ can one compute*

$$H^1(\mathcal{I}^p_{g,b}, \mathrm{End}(V))^{\mathrm{Sp}_{2g}(\mathbb{Z})}?$$

A broad vanishing theorem here would directly enlarge the known range of Litt's conjecture.

Question 7.3 (Beyond pure central parity)**.** *Can theorem 5.1 be extended to algebraic symplectic modules B with mixed $-I$-parity? The proof here breaks exactly because $\mathrm{End}(B)$ can then contain odd symplectic constituents, so the parity-density lemma no longer gives vanishing.*

Acknowledgments

This note is intended as a compact record of consequences of theorems of Johnson, Hain, Putman, Starkov, and Ershov–He, together with the tensor observation in theorem 5.1. The author thanks the mathematical community whose work is surveyed and combined here.

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