EXCEPTIONAL SHIMURA LOCAL SYSTEMS: MINIMAL-RANK RECOGNITION AND GEOMETRIC PROBES

WORKING DRAFT

ABSTRACT. Let $S = \mathrm{Sh}_K(G,X)^0(\mathbf{C})$ be a connected neat Shimura variety whose non-compact real adjoint factor is one of the exceptional Hermitian groups $E_{6(-14)}$ or $E_{7(-25)}$. We develop a general recognition framework for tautological local systems on S. The framework is independent of any particular coordinate calculation and is designed to reduce the problem of geometric origin to verifiable representation-theoretic and infinitesimal-period tests.

The first result is a minimal-rank recognition theorem. On an adjoint E_6 or E_7 quotient, every semisimple complex local system of rank below the adjoint representation has relatively compact monodromy; every non-precompact semisimple local system of adjoint rank is the adjoint tautological local system, up to finite character and outer automorphism. On Shimura towers whose derived group admits the minuscule representation, the analogous statements hold in ranks 27 for E_6 and 56 for E_7. Consequently any non-isotrivial geometric variation of the corresponding minimal rank is automatically the desired exceptional tautological local system as a Betti local system.

The second result is a probe-recognition theorem. Let ρ be an algebraic representation of the exceptional group and let $\{(H_i, X_i)\}$ be Hodge-type Shimura subdata which are ρ-determining in the sense that restriction to the H_i distinguishes ρ among the relevant algebraic representations. Then a semisimple local system on S whose restrictions to the S_{H_i} agree with the corresponding tautological restrictions is globally tautological. Since tautological systems restricted to Hodge-type subvarieties are of geometric origin, this gives a precise finite-probe strategy for testing any proposed global geometric construction.

We also record three structural consequences: a Tannakian reduction showing that one faithful geometric tautological system implies all tautological systems are geometric; a no-go theorem ruling out homogeneous weight-one/Siegel tensor-generator constructions in the exceptional cases; and an infinitesimal Mumford-Tate criterion showing how the first Higgs field and characteristic forms force exceptional monodromy. For E_7 the minimal target is a rank-56 Calabi-Yau type system with Hodge numbers $(1, 27, 27, 1)$; for E_6 it is a rank-27 complex weight-two system of type $(1, 16, 10)$, together with its conjugate for rational coefficients. The paper does not construct the missing family over S; it gives rank-only, probe, and infinitesimal criteria under which any such family is automatically the exceptional tautological one.

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1. Introduction

Shimura varieties of Hodge type carry universal abelian schemes. Their tautological variations of Hodge structure are therefore visibly geometric: the standard representation is realized in R^1 of the universal abelian scheme, and every other algebraic representation is obtained by tensor operations, duals, and direct summands. The exceptional Hermitian Shimura varieties are not of this kind. Deligne's classification excludes simple Shimura data of type E_6 and E_7 from abelian type; Lan's exposition states explicitly that the E_6 and E_7 cases are never of abelian type [13, Sec. 5.2.2]. Milne's motivic interpretation of Shimura varieties similarly removes factors of type E_6, E_7, and certain D-types from the main abelian-motivic construction [18].

Let

$$S = \mathrm{Sh}_K(G, X)^0(\mathbf{C}) \simeq \Gamma\backslash D$$

be a connected component at neat level. The two exceptional irreducible Hermitian domains are

$$D_{E_6} = E_{6(-14)}/(\mathrm{Spin}(10)\cdot U(1)), \qquad D_{E_7} = E_{7(-25)}/(E_6\cdot U(1)).$$

For an algebraic representation ρ of the relevant algebraic group, or of a central cover when the representation does not descend to the adjoint group, there is a tautological local system $\mathbb{V}_\rho$ on the corresponding connected Shimura tower. The motivating question is whether these systems are of geometric origin.

Question 1.1. After passing to finite etale level, is every exceptional tautological local system $\mathbb{V}_\rho$ a subquotient, direct summand, or tensor construction in systems $R^i f_*\mathbf{Q}(j)$ for smooth projective morphisms $f : Y \to S$?

The expected answer is yes. One conceptual reason is rigidity. Starkov's theorem gives a strong first-cohomology vanishing statement for lattices with no quotient lattice in a real hyperbolic or complex hyperbolic group [26]. It applies to pure $E_{6(-14)}$ and $E_{7(-25)}$ quotients. Hence all finite-dimensional semisimple local systems on such quotients are cohomologically rigid. Simpson's philosophy predicts that rigid integral local systems should be motivic or geometric; Langer and Simpson prove such a statement in rank 3 over smooth projective varieties [14], but the exceptional tautological ranks are much larger.

There is also strong arithmetic evidence. Diao, Lan, Liu, and Zhu construct a p-adic Riemann-Hilbert functor and prove compatibility with classical automorphic bundles for Shimura local systems attached to algebraic representations [4]. Klevdal and Patrikis prove independence of ℓ for canonical adjoint ℓ-adic local systems in the superrigid, non-abelian-type setting [12], and Patrikis proves the corresponding compatibility for the $G(\mathbf{Q}_\ell)$-valued systems themselves [19]. These theorems give exactly the arithmetic behavior expected of geometric systems, but they do not construct a smooth projective family over S realizing the Betti local system.

The purpose of this paper is to rebuild the reduction strategy in a form that is independent of explicit coordinate computations. The guiding idea is this: in the exceptional setting, it is often unnecessary to compute full monodromy. Superrigidity and the low-dimensional representation theory of E_6 and E_7 force any non-isotrivial local system of the minimal possible rank to be tautological. More generally, a finite collection of classical Hodge-type subvarieties can be used as probes; if a proposed global local system restricts correctly to a representation-theoretically determining collection of probes, then it is globally tautological.

Main results. We state the main results informally here; the precise formulations appear in the body of the paper.

The first result is a rank-only recognition theorem. It is a direct but useful consequence of Margulis superrigidity and the smallest-degree representation theory of exceptional groups.

Theorem (Minimal-rank recognition)**.** *Let $S = \Gamma\backslash D$ be a neat connected arithmetic quotient of a single exceptional Hermitian factor.*

(a) *If $D = D_{E_6}$ and the lattice is adjoint, then every semisimple complex local system of rank < 78 has relatively compact monodromy. Every semisimple non-precompact local system of rank 78 is the adjoint tautological system, up to finite character and the diagram involution.*
(b) *If $D = D_{E_7}$ and the lattice is adjoint, then every semisimple complex local system of rank < 133 has relatively compact monodromy. Every semisimple non-precompact local system of rank 133 is the adjoint tautological system, up to finite character.*
(c) *On connected Shimura towers whose derived group admits the minuscule representation, the corresponding minimal ranks are 27 for E_6 and 56 for E_7. Thus a non-precompact rank-27 system on the E_6 minuscule tower is one of the two minuscule tautological systems, and a non-precompact rank-56 system on the E_7 minuscule tower is the minuscule tautological system.*

The geometric consequence is especially simple: a non-isotrivial polarizable complex variation of Hodge structure of one of these minimal ranks is automatically tautological as a Betti local system. Thus, to construct the minuscule E_7 tautological system geometrically, it would be enough to construct over the E_7 Shimura variety any non-isotrivial rank-56 geometric summand with the correct central character; no exceptional monodromy calculation would remain. Similarly, for E_6 one seeks a non-isotrivial rank-27 complex summand, or a rank-54 rational system whose complexification is a minuscule-conjugate pair.

The second result abstracts the role of explicit restriction computations. A family of subgroups $H_i \subset G$ is called ρ-determining if the restrictions $\rho|_{H_i}$ distinguish ρ from the other algebraic representations which can occur in the relevant rank.

Theorem (Probe recognition)**.** *Let ρ be an algebraic representation giving a tautological system $\mathbb{V}_\rho$ on an exceptional Shimura variety S. Let $(H_i, X_i) \subset (G, X)$ be Hodge-type Shimura subdata which are ρ-determining. Let $\mathbb{L}$ be a semisimple local system on a finite cover of S. Suppose that, after passing to finite covers of the S_{H_i}, the restrictions $\mathbb{L}|_{S_{H_i}}$ and $\mathbb{V}_\rho|_{S_{H_i}}$ have the same semisimplification for all i. If $\mathbb{L}$ has non-precompact monodromy, then $\mathbb{L} \simeq \mathbb{V}_\rho$ up to finite character and the finite central/outer ambiguity.*

The theorem is useful because each $\mathbb{V}_\rho|_{S_{H_i}}$ is geometric when (H_i, X_i) is of Hodge type. It does not by itself extend the families on the probes to a family over S; rather, it says that any proposed global geometric local system can be checked on a finite list of classical subvarieties.

The third result is an infinitesimal version. Let $\mathbb{W}$ be a polarizable variation of Hodge structure over a finite cover of S. At a Hodge-generic point, the differential of the period map is a subspace of the $(-1, 1)$-part of the generic Mumford-Tate Lie algebra. If that subspace and its polarization adjoint generate the exceptional Lie algebra in the given representation, then the generic Mumford-Tate group contains the exceptional group. Combining this with superrigidity gives a monodromy criterion for recognizing tautological systems. The point is general: the characteristic forms computed in concrete models are not separate inputs to the paper; they are instances of this representation-theoretic "Lie algebra generated by the Higgs field" criterion.

Finally, we retain two structural reductions from the earlier note. First, one faithful geometric tautological system implies all tautological systems are geometric, by Tannakian generation. Second, a homogeneous weight-one/Siegel tensor-generator construction for a faithful exceptional tautological system would force the Shimura datum to be of Hodge type and hence abelian type, contradicting Deligne's classification. This rules out the standard universal-abelian strategy, while leaving open genuinely higher-cohomological constructions such as Calabi-Yau, Fano, hyperkahler, or covering constructions.

What is new and what is not. The vanishing of $H^1(\Gamma, M)$ and finite reduced Betti moduli are not new; they are standard consequences of Starkov's theorem and Weil's local-rigidity criterion. The superrigidity input is Margulis's theorem. The new contribution here is a general recognition package tailored to the exceptional geometric-origin problem: minimal-rank recognition, finite-probe recognition, and infinitesimal Mumford-Tate forcing, together with their consequences for possible geometric constructions. These results do not solve Theorem 1.1; they sharply constrain what a solution must look like.

Acknowledgements and status. This is a working draft. It is written as a reduction and recognition paper, not as a construction of the missing exceptional families. All uses of rigidity and superrigidity are standard; the representation-theoretic conclusions are recorded explicitly because they are useful for testing future constructions.

2. Set-up and central covers

2.1. **Exceptional quotients.** Throughout, L denotes one of the connected real Lie groups with adjoint group

$$E^{\mathrm{ad}}_{6(-14)} \qquad \text{or} \qquad E^{\mathrm{ad}}_{7(-25)}.$$

Let $D = L/K$ be the associated Hermitian symmetric domain. Let $\Gamma \subset L$ be a torsion-free arithmetic lattice, so that

$$S = \Gamma\backslash D$$

is a connected smooth complex algebraic variety by the Baily-Borel theorem and the theory of arithmetic quotients. In Shimura-theoretic applications, Γ is the fundamental group of a connected component of $\mathrm{Sh}_K(G,X)(\mathbf{C})$ at neat level.

We often distinguish two situations.

(i) The *adjoint tower*: local systems are attached to representations of the adjoint group. The smallest non-trivial representations are the adjoint representations of dimensions 78 for E_6 and 133 for E_7.

(ii) A *minuscule tower*: the Shimura datum is replaced, if necessary, by a central-isogenous datum whose derived group admits the minuscule representation. The relevant dimensions are 27 for E_6 and 56 for E_7.

The distinction matters. The 27-dimensional representations of E_6 and the 56-dimensional representation of E_7 do not descend to the adjoint group. They define honest local systems only on a Shimura tower whose connected derived group and level allow the corresponding central character. All statements below should be read on the appropriate connected tower. Passing between central-isogenous Shimura data introduces only finite central and finite-character ambiguities, but those ambiguities should not be suppressed.

2.2. **Tautological local systems.** Let $\rho : G \to \mathrm{GL}(V)$ be an algebraic representation defined over a field $F \subset \mathbf{C}$. On a connected component $S = \Gamma\backslash D$ for which ρ is defined on the relevant arithmetic group, the associated tautological F-local system is

$$\mathbb{V}_\rho = \Gamma\backslash(D \times V).$$

When ρ is a Hodge representation for the Shimura cocharacter, $\mathbb{V}_\rho$ underlies a homogeneous polarizable variation of Hodge structure. For the exceptional minuscule representations, the complex Hodge numbers are

$$E_6: \quad (1,16,10) \quad \text{for } \mathbf{27},$$

with dual ordering for $\mathbf{27}^\vee$, and

$$E_7: \quad (1,27,27,1) \quad \text{for } \mathbf{56}.$$

The adjoint representations have Hodge numbers

$$E_6: \quad (16,46,16), \qquad E_7: \quad (27,79,27).$$

These numbers follow from the decompositions of the complexified Lie algebras under the compact isotropy groups

$$E_6 \supset \mathrm{Spin}(10)\cdot \mathbf{G}_m, \qquad E_7 \supset E_6 \cdot \mathbf{G}_m.$$

2.3. **Geometric origin.**

Definition 2.1. Let S be a smooth connected complex quasi-projective variety. A semisimple F-local system $\mathbb{L}$ on S is *strongly of geometric origin* if, after passing to a finite etale cover $S' \to S$, the pullback of $\mathbb{L}$ belongs to the smallest semisimple Tannakian subcategory of F-local systems on S' containing all local systems

$$R^i f_* F(j)$$

for smooth projective morphisms $f : Y \to S'$. It is *weakly of geometric origin* if the same condition holds after replacing S' by a Zariski open dense subset.

The distinction is important. Simpson-type conjectures and the Langer-Simpson theorem naturally use a weak notion: one may remove a divisor before realizing the local system by a smooth projective family. The original Shimura-motivic question is often stronger and asks for a realization over the Shimura variety itself, or over a controlled toroidal compactification. Unless

explicitly stated otherwise, this paper proves recognition statements for Betti local systems; it does not assert that weak geometric origin automatically upgrades to strong geometric origin.

3. Rigidity inputs

This section records standard consequences of Starkov's vanishing theorem. They are included to fix conventions, not as new results.

Definition 3.1. A pair (L, Γ) consisting of a connected semisimple real Lie group and a lattice satisfies the *Starkov condition* if there is no epimorphism

$$L \longrightarrow H$$

onto a Lie group H locally isomorphic to $\mathrm{SO}(1,n)$ or $\mathrm{SU}(1,n)$ such that the image of Γ is a lattice in H.

Theorem 3.2 (Starkov)**.** *If (L, Γ) satisfies the Starkov condition, then*

$$H^1(\Gamma, M) = 0$$

for every finite-dimensional real representation M of Γ.

For $L = E^{\mathrm{ad}}_{6(-14)}$ or $E^{\mathrm{ad}}_{7(-25)}$, the condition is automatic: neither group has a quotient locally isomorphic to $\mathrm{SO}(1,n)$ or $\mathrm{SU}(1,n)$.

Corollary 3.3 (Cohomological rigidity)**.** *Let $S = \Gamma\backslash D$ be a pure exceptional quotient as above. For every semisimple complex local system $\mathbb{L}$ on S,*

$$H^1(S, \operatorname{End}\mathbb{L}) = 0.$$

Equivalently, every semisimple representation $\Gamma \to \mathrm{GL}_n(\mathbf{C})$ is infinitesimally rigid.

Proof. Since D is contractible, $H^1(S, \operatorname{End}\mathbb{L}) = H^1(\Gamma, \operatorname{End} V)$ for the corresponding representation of Γ. Apply Theorem 3.2 to the underlying real representation. □

Corollary 3.4 (Finite reduced Betti moduli)**.** *For every reductive algebraic group $H/\mathbf{Q}$, the reduced affine character variety*

$$\mathsf{M}_B(\Gamma, H)_{\mathrm{red}} := \big(\mathrm{Hom}(\Gamma, H) /\!\!/ H\big)_{\mathrm{red}}$$

is a finite reduced scheme over $\mathbf{Q}$. In particular, for each fixed n there are only finitely many isomorphism classes of semisimple rank-n complex local systems on S.

Proof. By Weil's local-rigidity criterion, the vanishing of $H^1(\Gamma, \mathfrak{h}_{\mathrm{Ad}\,\sigma})$ at a semisimple representation $\sigma : \Gamma \to H(\mathbf{C})$ implies that the corresponding point of the reduced character variety is isolated. The representation variety of a finitely generated group into an affine algebraic group is of finite type, hence the reduced GIT quotient has only finitely many isolated closed points in each connected component and therefore is finite. Since the construction is defined over $\mathbf{Q}$, the reduced finite scheme is finite over $\mathbf{Q}$. □

Remark 3.5. The preceding corollary is a standard finite-representation-type consequence of rigidity; it should not be advertised as a new theorem. It is nevertheless useful because it gives finite certificates for recognizing a fixed tautological local system.

Corollary 3.6 (Finite trace certificates)**.** *Fix a semisimple representation $\rho_0 : \Gamma \to \mathrm{GL}_n(\mathbf{C})$. There exist finitely many elements $\gamma_1, \dots, \gamma_N \in \Gamma$ such that any semisimple representation $\rho : \Gamma \to \mathrm{GL}_n(\mathbf{C})$ satisfying*

$$\mathrm{Char}_{\rho(\gamma_i)}(T) = \mathrm{Char}_{\rho_0(\gamma_i)}(T) \quad \textit{for all } i$$

is isomorphic to ρ_0.

Proof. The reduced rank-n character variety is finite. The closed point corresponding to ρ_0 is cut out by finitely many regular functions. By Procesi's trace generation theorem for GL_n-characters, these functions may be chosen among coefficients of characteristic polynomials of finitely many group elements. This gives the required finite list. □

4. Minimal-rank recognition

We now prove the rank-only recognition theorem. This is the first place where exceptional representation theory enters.

4.1. **Small representations of E_6 and E_7.** Let $\mathsf{E}_6^{\mathrm{sc}}$ and $\mathsf{E}_7^{\mathrm{sc}}$ denote the simply connected complex groups of types E_6 and E_7.

Lemma 4.1 (Small complex representations)**.** *The following are the non-trivial irreducible complex representations of smallest dimension.*

(a) *For $\mathsf{E}_6^{\mathrm{sc}}$, the smallest non-trivial irreducibles have dimensions*

$$27,\ 27,\ 78,\ 351,\dots.$$

The two representations of dimension 27 are dual minuscule representations, and the representation of dimension 78 is the adjoint representation.

(b) *For $\mathsf{E}_7^{\mathrm{sc}}$, the smallest non-trivial irreducibles have dimensions*

$$56,\ 133,\ 912,\dots.$$

The representation of dimension 56 is minuscule, and the representation of dimension 133 is adjoint.

(c) *For the adjoint complex groups, the minuscule representations in (a) and (b) do not descend. Thus the smallest non-trivial irreducible representations of the adjoint groups have dimensions 78 for E_6^{ad} and 133 for E_7^{ad}.*

Proof. Parts (a) and (b) are the standard Weyl-dimension calculation for the fundamental representations of E_6 and E_7; see, for example, the tables of McKay-Patera or Luebeck's tables of small-degree representations. Part (c) follows because a representation descends to the adjoint group precisely when its highest weight lies in the root lattice. The minuscule highest weights of dimensions 27 and 56 do not lie in the root lattice, whereas the adjoint highest root does. □

4.2. **Superrigidity formulation.** We use the following standard form of Margulis superrigidity.

Theorem 4.2 (Margulis superrigidity, representation-theoretic form)**.** *Let Γ be an irreducible arithmetic lattice in a connected semisimple real Lie group L with no compact factors and real rank at least 2. Let*

$$\sigma : \Gamma \longrightarrow \mathrm{GL}(V_{\mathbf{C}})$$

be a semisimple representation whose image is not relatively compact. After replacing Γ by a finite-index subgroup, σ is the product of a representation with relatively compact image centralizing the algebraic factor and the restriction to Γ of a continuous homomorphism

$$\widetilde{\sigma} : L \longrightarrow \mathrm{GL}(V_{\mathbf{C}}).$$

The complexified differential of $\widetilde{\sigma}$ is an algebraic representation of $\mathrm{Lie}(L)_{\mathbf{C}}$.

For a $\mathbf{Q}$-simple exceptional Shimura datum with a single non-compact real factor $E_{6(-14)}$ or $E_{7(-25)}$, the hypotheses hold. For products, the same conclusions apply after decomposing into irreducible factors; the statements below are written for a single factor to keep notation uncluttered.

Theorem 4.3 (Minimal-rank recognition: adjoint towers)**.** *Let $S = \Gamma\backslash D$ be a neat connected arithmetic quotient of the adjoint exceptional Hermitian domain.*

(a) *If $D = D_{E_6}$, every semisimple complex local system of rank < 78 has relatively compact monodromy. Every semisimple complex local system of rank 78 with non-precompact monodromy is irreducible and is isomorphic to the adjoint tautological local system, up to finite character and the outer diagram involution.*

(b) *If $D = D_{E_7}$, every semisimple complex local system of rank < 133 has relatively compact monodromy. Every semisimple complex local system of rank 133 with non-precompact monodromy is irreducible and is isomorphic to the adjoint tautological local system, up to finite character.*

Proof. Let $\mathbb{L}$ be a semisimple local system and write its monodromy as a direct sum of irreducible representations of Γ. If one irreducible summand has non-precompact image, then by Theorem 4.2 it extends, after finite index and up to a relatively compact centralizing factor, to a non-trivial algebraic representation of the adjoint exceptional group. By Theorem 4.1, the dimension of such a summand is at least 78 for adjoint E_6 and at least 133 for adjoint E_7. This proves precompact monodromy below the stated rank.

At the stated rank, the existence of any non-precompact summand forces that summand to have the whole rank and hence $\mathbb{L}$ is irreducible. The same small-representation lemma identifies the extension with the adjoint representation. The only nontrivial outer automorphism in these two cases is the diagram involution of E_6. □

Theorem 4.4 (Minimal-rank recognition: minuscule towers)**.** *Work on a connected Shimura tower whose derived group admits the indicated minuscule representation.*

(a) *In type E_6, every semisimple complex local system of rank < 27 has relatively compact monodromy. Every semisimple complex local system of rank 27 with non-precompact monodromy is one of the two minuscule tautological systems $\mathbf{27}$ or $\mathbf{27}^\vee$, up to finite character and diagram involution.*

(b) *In type E_7, every semisimple complex local system of rank < 56 has relatively compact monodromy. Every semisimple complex local system of rank 56 with non-precompact monodromy is the minuscule tautological system $\mathbf{56}$, up to finite character.*

Proof. The proof is identical to that of Theorem 4.3, using the simply connected small-representation bounds in Theorem 4.1. The central character determines on which central-isogenous tower the representation is an honest local system. □

Corollary 4.5 (Rank-only geometric criterion)**.** *Let S be as in Theorem 4.4 or Theorem 4.3. Let $\mathbb{W}$ be a polarizable complex variation of Hodge structure over S whose underlying local system has one of the minimal ranks $27, 56, 78, 133$ in the corresponding tower. If the period map of $\mathbb{W}$ is nonconstant, then the underlying Betti local system is the corresponding tautological local system, up to finite character and the stated finite ambiguities.*

Proof. A polarizable complex variation of Hodge structure with relatively compact monodromy gives a unitary flat bundle and hence a zero Higgs field for a polarizable variation of Hodge structure. Thus the period map is constant. Therefore nonconstant period map implies non-precompact monodromy. Apply Theorems 4.3 and 4.4. □

Remark 4.6. The E_6 minuscule representation is complex. A rational polarized variation will normally contain both conjugate minuscule pieces. Thus the rank-only criterion over $\mathbf{Q}$ should be applied either after extending coefficients to a number field, where a rank-27 summand appears, or to a rank-54 rational system whose complexification is $\mathbf{27} \oplus \mathbf{27}^\vee$, together with possible finite-character twists.

5. Tannakian reduction and the weight-one no-go theorem

5.1. **One faithful system suffices.** Let $\mathsf{GO}(S)$ be the semisimple Tannakian subcategory of local systems generated by systems of geometric origin in the sense of Theorem 2.1. It is closed under direct sums, subquotients, tensor products, duals, and finite pullback.

Proposition 5.1 (Faithful tensor generator)**.** *Let $\rho_0 : G \to \mathrm{GL}(W)$ be a faithful algebraic representation on a connected Shimura tower. If the tautological local system $\mathbb{V}_{\rho_0}$ is of geometric origin, then $\mathbb{V}_\rho$ is of geometric origin for every algebraic representation ρ of G defined on the same tower.*

Proof. Since ρ_0 is faithful, the representation $W \oplus W^\vee$ tensor-generates $\mathrm{Rep}(G)$: every irreducible algebraic representation of G is a subquotient of a finite direct sum of tensor powers of $W \oplus W^\vee$. The tautological construction is a tensor functor from $\mathrm{Rep}(G)$ to local systems on S, and geometric-origin local systems form a Tannakian subcategory. Hence all $\mathbb{V}_\rho$ are of geometric origin. □

For the adjoint tower, the adjoint representation is faithful. For central covers, a minuscule representation is often faithful modulo a finite center; adding a finite central character or its dual handles the residual center.

5.2. **No homogeneous Siegel tensor-generator construction.** The following result clarifies what the non-abelian-type obstruction does and does not say.

Proposition 5.2 (No homogeneous weight-one tensor-generator shortcut)**.** *Let (G, X) be a simple exceptional Shimura datum of type E_6 or E_7. There is no faithful symplectic representation*

$$G \longrightarrow \mathrm{GSp}(W, \psi)$$

which makes (G, X) a Shimura subdatum of a Siegel Shimura datum. Consequently, a faithful exceptional tautological local system cannot be geometrized by the standard homogeneous construction from R^1 of a universal abelian scheme over S.

Proof. A faithful symplectic representation realizing (G, X) inside a Siegel datum is precisely a Hodge-type realization. Hodge-type Shimura data are of abelian type. Deligne's classification, as recalled for example in [13, Sec. 5.2.2], says that simple E_6 and E_7 Hermitian Shimura data are never of abelian type. This gives the contradiction. □

Remark 5.3. Theorem 5.2 rules out only the homogeneous universal-abelian tensor-generator strategy. It does not rule out higher-cohomological constructions. A successful construction may still use Calabi-Yau varieties, Fano varieties, hyperkahler varieties, covers of auxiliary parameter spaces, or other non-abelian motives.

6. Superrigidity recognition

This section gives a general recognition theorem that applies once a candidate geometric local system has been constructed.

Definition 6.1. Let L be an exceptional real Hermitian group and let $\rho_0 : L^\sharp \to \mathrm{GL}(V)$ be an algebraic representation of a central cover. A semisimple representation

$$\sigma : \Gamma' \to \mathrm{GL}(V)$$

of a finite-index subgroup is said to have *model type* $(L^\sharp, \rho_0)$ if its image is non-precompact and the complexified algebraic representation obtained from Margulis superrigidity has the same highest weight as ρ_0, up to an automorphism of the complex exceptional group and a character trivial on the derived group.

Theorem 6.2 (Local-system recognition)**.** *Let $S = \Gamma\backslash D$ be a pure exceptional quotient and let ρ_0 be an absolutely irreducible algebraic representation defining a tautological local system $\mathbb{V}_{\rho_0}$ on a connected central-isogenous tower. Let*

$$\mathbb{L}$$

be a semisimple complex local system on a finite etale cover S' of S. If the monodromy representation of $\mathbb{L}$ has model type $(L^\sharp, \rho_0)$, then after replacing S' by a further finite cover,

$$\mathbb{L} \simeq \mathbb{V}_{\rho_0} \otimes \chi,$$

where χ is a finite character. The only additional ambiguity is the finite group of diagram automorphisms and central characters compatible with the chosen tower.

Proof. Let $\Gamma' = \pi_1(S')$. By Margulis superrigidity, after finite index the monodromy representation $\sigma : \Gamma' \to \mathrm{GL}(V)$ is the product of a relatively compact representation centralizing the algebraic factor and the restriction of a continuous homomorphism $\widetilde{\sigma} : L \to \mathrm{GL}(V)$. By the model-type hypothesis, the complexified differential of $\widetilde{\sigma}$ is conjugate to $d\rho_0 \circ d\alpha$ for an automorphism α of the complex exceptional Lie algebra preserving the relevant real form. Thus the two representations agree on the derived group after applying the corresponding outer automorphism. In the irreducible model-type situation, Schur's lemma makes the centralizing factor scalar. Since higher-rank arithmetic lattices are finitely generated and have finite abelianization after passing to finite index, this scalar character has finite image. This proves the claim. □

Corollary 6.3 (Construction criterion)**.** *Let $f : Y \to S'$ be a smooth projective morphism over a finite etale cover of a pure exceptional Shimura variety. Suppose an absolutely irreducible semisimple direct summand*

$$\mathbb{W} \subset R^i f_* \mathbf{C}(j)$$

has model type $(L^\sharp, \rho_0)$. Then $\mathbb{W}$ is the corresponding tautological local system, up to finite character and the finite central/outer ambiguity. In particular, if ρ_0 is faithful, then all tautological local systems on the same tower are of geometric origin.

Proof. Apply Theorem 6.2 to $\mathbb{W}$, then Theorem 5.1 for the final sentence. □

The minimal-rank theorem is a way to verify the model-type hypothesis without computing monodromy: in the minimal ranks, non-isotriviality already forces the model type.

7. Probe recognition

Explicit computations often arise by restricting an exceptional representation to a classical subgroup. This section abstracts that idea into a general theorem.

7.1. **Determining families of subgroups.** Let G be a connected complex semisimple group and let ρ be a finite-dimensional semisimple representation of G. Let $\mathsf{H}_1, \dots, \mathsf{H}_r \subset \mathsf{G}$ be connected reductive subgroups.

Definition 7.1. The family $\{\mathsf{H}_i\}$ is *ρ-determining in rank n* if for every semisimple representation τ of G of dimension $n = \dim \rho$,

$$\tau|_{\mathsf{H}_i} \simeq \rho|_{\mathsf{H}_i} \quad \text{for all } i$$

implies

$$\tau \simeq \rho \circ \alpha$$

for some automorphism α of G which acts trivially on the prescribed collection of restrictions. For a finite set $\mathcal{R}$ of representations, the same definition with $\tau \in \mathcal{R}$ gives *ρ-determining in $\mathcal{R}$*.

This definition deliberately separates the representation-theoretic verification from the geometric argument. In applications, one checks the condition by branching rules, by highest weights, or by character computations. The computational supplement to this project gives coordinate-level tests for the minuscule cases, but the theorem below does not use those coordinates.

Lemma 7.2. *If the restrictions of characters*

$$\chi_\rho|_{\mathsf{H}_i}$$

distinguish ρ among all semisimple representations of dimension n that can occur as superrigidity extensions of local systems on S, then $\{\mathsf{H}_i\}$ is ρ-determining in rank n.

Proof. For reductive groups over $\mathbf{C}$, a semisimple representation is determined by its character. Thus equality of the restricted characters on all H_i is exactly equality of the restrictions as semisimple representations. If this data distinguishes ρ in the relevant finite set, the conclusion follows. □

7.2. **Hodge-type probes.** Let (G, X) be an exceptional Shimura datum and let

$$(H_i, X_i) \subset (G, X)$$

be Shimura subdata. Write S_i for the corresponding connected Shimura subvarieties, after choosing compatible neat levels.

Definition 7.3. A family $\{(H_i, X_i)\}$ is a *geometric probe system for ρ* if

(i) each (H_i, X_i) is of Hodge type, and
(ii) the complex groups $H_{i,\mathbf{C}}^{\mathrm{der}} \subset G_{\mathbf{C}}^{\mathrm{der}}$ form a ρ-determining family in the sense of Theorem 7.1.

The word "geometric" refers to the fact that restrictions to Hodge-type subvarieties are realized by universal abelian schemes.

Proposition 7.4 (Geometricity on Hodge-type probes)**.** *Let $(H, X_H) \subset (G, X)$ be a Hodge-type Shimura subdatum and let*

$$i : S_H \longrightarrow S_G$$

be the induced morphism on connected neat Shimura varieties. For every algebraic representation ρ of G,

$$i^*\mathbb{V}_\rho$$

is of geometric origin on S_H, after finite etale cover and the usual central-character bookkeeping.

Proof. Since (H, X_H) is of Hodge type, after passing to sufficiently small level there is a universal abelian scheme $\pi : A \to S_H$ realizing a faithful symplectic representation of H in $R^1\pi_*\mathbf{Q}$. The restricted representation $\rho|_H$ belongs to the Tannakian category generated by this faithful representation and its dual. Hence the corresponding local system is in the Tannakian category generated by cohomology of powers of A. □

Theorem 7.5 (Probe recognition)**.** *Let S be a pure exceptional Shimura variety and let ρ be an absolutely irreducible algebraic representation defining a tautological local system $\mathbb{V}_\rho$. Let $\{(H_i, X_i)\}$ be a geometric probe system for ρ. Let $\mathbb{L}$ be a semisimple complex local system on a finite etale cover S' of S with non-precompact monodromy. Suppose that for every i, after passing to a common finite cover of the corresponding probe subvariety,*

$$\mathbb{L}|_{S_i} \simeq \mathbb{V}_\rho|_{S_i}$$

as semisimple local systems. Then

$$\mathbb{L} \simeq \mathbb{V}_\rho \otimes \chi$$

up to finite character and the finite central/outer ambiguity allowed by the chosen tower.

Remark 7.6. The irreducibility hypothesis is used only to make the compact centralizing factor in superrigidity scalar, hence finite after passing to finite index. For reducible representations one should either apply the theorem constituentwise or include, as an additional hypothesis, that the compact centralizing error is finite.

Proof. By Margulis superrigidity, the monodromy representation of $\mathbb{L}$ extends, after finite index and up to a relatively compact centralizing factor, to an algebraic representation τ of the complexified exceptional group. The restriction hypothesis implies that

$$\tau|_{H_{i,\mathbf{C}}^{\mathrm{der}}} \simeq \rho|_{H_{i,\mathbf{C}}^{\mathrm{der}}}$$

for all i, because the arithmetic groups in the H_i are Zariski dense. Since the probes are ρ-determining, $\tau \simeq \rho$ up to the specified automorphism. The remaining difference on the lattice is a finite character, as in Theorem 6.2. □

Corollary 7.7 (Finite-probe construction criterion)**.** *Let $f : Y \to S'$ be a smooth projective family and let $\mathbb{W} \subset R^i f_* \mathbf{C}(j)$ be an absolutely irreducible semisimple direct summand with non-precompact monodromy. If the restrictions of $\mathbb{W}$ to a geometric probe system for ρ agree with the corresponding tautological restrictions, then $\mathbb{W}$ is the tautological local system $\mathbb{V}_\rho$, up to finite character and finite central/outer ambiguity.*

Remark 7.8. Theorem 7.7 is one of the main practical outputs of the paper. It says that a proposed global geometric construction need not come with a complete exceptional monodromy calculation. It is enough to check its restrictions to a finite list of classical Hodge-type probes which determine the representation.

8. Infinitesimal Mumford-Tate forcing

The previous criteria use rank or restrictions. This section gives an intrinsic Hodge-theoretic criterion. It abstracts the role of characteristic forms such as the E_7 cubic and the E_6 quadratic system without using their coordinate equations.

8.1. **The Higgs-generated Lie algebra.** Let $\mathbb{W}$ be a polarizable variation of Hodge structure with coefficients in a number field F, and fix an embedding $F \hookrightarrow \mathbf{C}$. Work after this extension of scalars. Let U be a smooth connected complex variety. Let $u \in U$ be a Hodge-generic point, let $W = \mathbb{W}_u$, and let M be the generic Mumford-Tate group. The Hodge decomposition of W induces a Hodge decomposition of $\mathfrak{m} = \mathrm{Lie}(M_{\mathbf{C}})$. The differential of the period map gives a linear map

$$\theta_u : T_{U,u} \longrightarrow \bigoplus_p \mathrm{Hom}(W^{p,q}, W^{p-1,q+1}).$$

Equivalently, using the infinitesimal action of $\mathfrak{m}$ on W, it is a subspace of $\mathfrak{m}^{-1,1}$.

Lemma 8.1 (Infinitesimal period maps lie in the generic MT algebra)**.** *At a Hodge-generic point u, the image of the infinitesimal period map is contained in $\mathfrak{m}^{-1,1} \subset \mathrm{End}(W)$.*

Proof. The period map of a variation with generic Mumford-Tate group M factors through the Mumford-Tate domain associated with M. The horizontal tangent space of this domain at a Hodge-generic point is $\mathfrak{m}^{-1,1}$. □

Let Q be a polarization. For a subspace $A \subset \mathrm{End}(W)$, write $A^\dagger$ for the subspace of Q-adjoints. Define

$$\mathfrak{l}(A) := \text{the Lie subalgebra of } \mathrm{End}(W) \text{ generated by } A + A^\dagger.$$

Proposition 8.2 (Higgs-generated lower bound)**.** *Let $\mathbb{W}$ be a polarizable variation of Hodge structure with number-field coefficients and let u be Hodge-generic. Put*

$$A = \operatorname{im}\theta_u \subset \operatorname{End}(W).$$

Then

$$\mathfrak{l}(A) \subset \operatorname{Lie}(M_{\mathbf{C}})^{\mathrm{der}}.$$

In particular, if $\mathfrak{l}(A)$ is isomorphic, in its action on W, to the image of an exceptional Lie algebra $\mathfrak{g}_{\mathbf{C}}$ under a representation ρ, then the generic Mumford-Tate group contains $\rho(G_{\mathbf{C}})$ in its derived group.

Proof. By Theorem 8.1, $A \subset \mathfrak{m}^{-1,1}$. Since M is reductive and the polarization identifies adjoint Hodge pieces, $A^\dagger \subset \mathfrak{m}^{1,-1}$. Therefore the Lie algebra generated by $A + A^\dagger$ is contained in $\mathfrak{m}_{\mathbf{C}}^{\mathrm{der}}$. □

8.2. **The homogeneous exceptional case.** For an irreducible Hermitian symmetric domain, the complexified Lie algebra has a three-step grading

$$\mathfrak{g}_{\mathbf{C}} = \mathfrak{g}^{-1,1} \oplus \mathfrak{g}^{0,0} \oplus \mathfrak{g}^{1,-1}$$

with $\mathfrak{g}^{-1,1}$ abelian and identified with the tangent space at the base point. The Lie algebra $\mathfrak{g}_{\mathbf{C}}$ is generated by $\mathfrak{g}^{-1,1} \oplus \mathfrak{g}^{1,-1}$.

Definition 8.3. Let $\rho : G_{\mathbf{C}} \to \mathrm{GL}(V)$ be a Hodge representation of an exceptional Hermitian group. Its *homogeneous Higgs symbol* is the subspace

$$A_\rho := \rho(\mathfrak{g}^{-1,1}) \subset \operatorname{End}(V).$$

A variation $\mathbb{W}$ *has the ρ-symbol at u* if, after identifying the Hodge-graded vector space $\operatorname{gr} W_u$ with the homogeneous Hodge-graded vector space V, the image of its infinitesimal period map is A_ρ.

Proposition 8.4 (Exceptional Lie algebra forced by the first Higgs field)**.** *Let ρ be a faithful Hodge representation of E_6 or E_7 on the relevant central tower. If a polarizable variation of Hodge structure with number-field coefficients $\mathbb{W}$ has the ρ-symbol at a Hodge-generic point u, then its generic Mumford-Tate group contains $\rho(G_{\mathbf{C}})$. If, in addition, the underlying local system has rank equal to one of the minimal ranks in Theorems 4.3 and 4.4 and nonconstant period map, then the underlying local system is tautological.*

Proof. For the homogeneous symbol, $A_\rho = \rho(\mathfrak{g}^{-1,1})$. Its polarization adjoint is $\rho(\mathfrak{g}^{1,-1})$. Since $\mathfrak{g}_{\mathbf{C}}$ is generated by $\mathfrak{g}^{-1,1} \oplus \mathfrak{g}^{1,-1}$, the Lie algebra generated by $A_\rho + A_\rho^\dagger$ is $\rho(\mathfrak{g}_{\mathbf{C}})$. The first assertion follows from Theorem 8.2. The final assertion follows from minimal-rank recognition: nonconstant period map implies non-precompact monodromy, and the only possible non-precompact local system in that rank is the corresponding tautological system. □

8.3. **Characteristic forms as coordinate-free tests.** The condition of having the full homogeneous symbol is often too strong to check directly. Characteristic forms give invariant tests for it. Let

$$E = \bigoplus_p E^p$$

be the Hodge bundle of a variation. Iterating the Higgs field gives maps

$$\theta^{(k)} : \operatorname{Sym}^k T_U \longrightarrow \bigoplus_p \operatorname{Hom}(E^p, E^{p-k}).$$

At a point, the kernels and images of these maps are the characteristic forms of the variation.

Criterion 8.5 (Characteristic-form test)**.** *Let ρ be a homogeneous Hodge representation of an exceptional Hermitian group. Suppose a polarizable variation $\mathbb{W}$ with number-field coefficients over S' has the same Hodge numbers as $\mathbb{V}_\rho$ and, at a Hodge-generic point, its characteristic forms are equivalent under the natural action of the product of the Hodge-graded automorphism groups to the homogeneous characteristic forms of ρ. If this equivalence implies equality of the first Higgs symbol A_ρ, then Theorem 8.4 applies.*

Remark 8.6. For E_7 in the minuscule representation, this criterion is closely related to the Gross-Sheng-Zuo-Friedman-Laza theory of Calabi-Yau type variations over Hermitian symmetric domains and to Robles's characteristic-form rigidity theorem for the tube-domain cases [8, 24, 5, 22]. For E_6, the analogous test is the spinorial second characteristic form. The present paper does not need coordinate equations for these tensors; it uses only the fact that the homogeneous tensors determine the exceptional Higgs symbol in the cases under consideration.

9. Exceptional construction targets

We now translate the general criteria into concrete but coordinate-free construction targets.

9.1. **The E_7 minuscule target.** On a minuscule E_7 tower, the smallest non-trivial tautological local system has rank 56 and Hodge numbers

$$(h^{3,0}, h^{2,1}, h^{1,2}, h^{0,3}) = (1, 27, 27, 1).$$

The tangent representation at the base point is the 27-dimensional minuscule representation of the compact isotropy group E_6. The homogeneous Yukawa cubic is the unique E_6-invariant cubic on this representation, equivalently the cubic norm on the exceptional Jordan algebra.

The rank-only recognition theorem gives the following criterion.

Corollary 9.1 (E_7 minimal geometric criterion)**.** *Let S' be a finite cover of the minuscule E_7 Shimura tower. Suppose there is a smooth projective morphism $f : Y \to S'$ and a semisimple direct summand*

$$\mathbb{W} \subset R^3 f_* \mathbf{C}(j)$$

of rank 56 with nonconstant period map. Then the underlying local system of $\mathbb{W}$ is the 56-dimensional tautological local system, up to finite character. If $\mathbb{W}$ is a faithful tensor generator, then all tautological systems on the same tower are of geometric origin.

Proof. Apply Theorem 4.5 and then Theorem 5.1. □

In practice one expects $j = 0$ and a Calabi-Yau type summand. The characteristic-form test says that if the Yukawa cubic is the homogeneous E_6-invariant cubic at a Hodge-generic point, then the variation has the expected exceptional infinitesimal period data. If the rank is 56, non-isotriviality already identifies the Betti local system.

9.2. **The E_6 minuscule target.** On a minuscule E_6 tower, the smallest complex tautological systems have rank 27 and Hodge numbers

$$(h^{2,0}, h^{1,1}, h^{0,2}) = (1, 16, 10)$$

for one minuscule representation, with dual Hodge numbers for the conjugate representation. Over $\mathbf{Q}$ one generally expects the pair $\mathbf{27} \oplus \mathbf{27}^\vee$.

Corollary 9.2 (E_6 minimal geometric criterion)**.** *Let S' be a finite cover of a minuscule E_6 Shimura tower. Suppose there is a smooth projective morphism $f : Y \to S'$ and, after extending coefficients to a number field F, a semisimple direct summand*

$$\mathbb{W} \subset R^2 f_* F(j)$$

of F-rank 27 with nonconstant period map. Then $\mathbb{W}$ is one of the two minuscule tautological systems, up to finite character and diagram involution. Equivalently, a rational rank-54 summand whose complexification is a non-isotrivial conjugate pair is the rational minuscule tautological pair.

Proof. Apply Theorem 4.5 in the minuscule E_6 case and use Theorem 4.6 for the rational formulation. □

The homogeneous second characteristic form is the unique spinorial quadratic map associated with the isotropy group Spin(10). Checking this form is a way to verify that a candidate family has the expected infinitesimal Hodge theory, but the rank-only criterion shows that once a non-isotrivial rank-27 summand exists on the correct tower, the underlying local system is already forced to be tautological.

9.3. **Adjoint targets.** If one works only on the adjoint Shimura variety, the minuscule systems need not be honest local systems. The universal targets are then the adjoint local systems.

Corollary 9.3 (Adjoint geometric criterion)**.** *Let S' be a finite cover of the adjoint exceptional Shimura variety. If a smooth projective family over S' has a semisimple direct summand of rank 78 in type E_6, respectively rank 133 in type E_7, with nonconstant period map, then that summand is the adjoint tautological local system, up to finite character and the E_6 diagram involution.*

Proof. This is Theorem 4.5 applied to Theorem 4.3. □

10. Relation with Simpson-type geometricity

The results above are recognition statements. They do not prove that the tautological systems are geometric. There is, however, a clean conditional implication from Simpson-type conjectures.

Conjecture 10.1 (Rigidity implies geometric origin, schematic form)**.** *Let U be a smooth complex quasi-projective variety. Every irreducible integral local system on U which is cohomologically rigid and has quasi-unipotent local monodromy at infinity is weakly of geometric origin.*

This is deliberately schematic. One must specify the coefficient field, integrality, boundary conditions, and whether the conclusion is weak or strong geometric origin. Langer and Simpson prove a rank-3 projective case [14].

Proposition 10.2 (Conditional geometricity)**.** *Assume a form of Theorem 10.1 applicable to the tautological local systems on a pure exceptional Shimura variety S. Then every irreducible tautological local system on S is weakly of geometric origin. If the conjecture is available in a strong-over-S form, then the conclusion is strong geometric origin.*

Proof. By Theorem 3.3, the local systems are cohomologically rigid. They are integral after choosing a stable lattice and passing to sufficiently small level. Boundary monodromy in toroidal compactifications is quasi-unipotent, and becomes unipotent after finite level. The conjecture applies. Reducible representations follow by semisimplicity and Tannakian closure. □

11. How the criteria would establish new cases

We finish by spelling out how the preceding theory should be used.

11.1. **Minimal-rank route.** For E_7, construct a smooth projective family $f : Y \to S'$ over the minuscule tower and find a rank-56 non-isotrivial direct summand of $R^3 f_* \mathbf{C}$. The rank-only theorem immediately identifies it with the 56-dimensional tautological local system. Hodge-theoretic checks such as Calabi-Yau type and the invariant cubic are not needed for the Betti recognition, but they are essential for finding and verifying the summand inside geometry.

For E_6, construct after coefficient extension a rank-27 non-isotrivial summand, or rationally a conjugate rank-54 pair. The rank-only theorem identifies the Betti local system with the minuscule tautological system. The spinorial second characteristic form is the expected infinitesimal Hodge-theoretic signature of the construction.

For adjoint Shimura varieties, use ranks 78 and 133.

11.2. **Probe route.** Instead of proving exceptional monodromy directly, construct a global geometric local system $\mathbb{W}$ and check it on a finite geometric probe system. The restrictions of the tautological system to Hodge-type probes are geometric by Theorem 7.4; hence such restrictions can often be described using universal abelian schemes, Kuga-Satake constructions, or unitary PEL data. If the probe restrictions determine the representation, Theorem 7.5 identifies $\mathbb{W}$ globally.

This route is useful when the expected summand has rank larger than the minimal rank, or when the geometric construction naturally decomposes on classical subdomains.

11.3. **Infinitesimal route.** Construct a polarizable variation over S' and compute its Higgs field at a Hodge-generic point. If the Higgs image and its adjoint generate the exceptional Lie algebra in the desired representation, then the generic Mumford-Tate group contains the exceptional group. Combined with superrigidity and rank constraints, this can force the underlying local system to be tautological. Characteristic forms package the same test invariantly: for E_7 the relevant tensor is the invariant cubic; for E_6 it is the spinorial quadratic system.

11.4. **What remains open.** The recognition criteria reduce the original problem to the construction of a suitable family. They do not supply such a family. The missing geometric input is one of the following:

(i) a non-isotrivial geometric summand of minimal exceptional rank over the appropriate tower;
(ii) a geometric summand whose restrictions to a determining system of Hodge-type probes are the tautological restrictions;
(iii) a Simpson-type theorem in the exceptional ranks;
(iv) or a direct motivic construction of the exceptional tautological systems.

Any of these would prove new cases of geometric origin. The results of this paper are designed to make the verification step as small as possible once a candidate family is found.

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