Sub-Markov chain certificates for weighted antichain bounds

A formal note motivated by Erdős problem #1196

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Abstract

Motivated by the recent forum discussion of Erdős problem #1196, and in particular by Terence Tao's question of how broadly the new Markov-chain idea can be used, we isolate a general "chain certificate" principle. The principle is simple: a sub-Markov kernel K together with a finite source measure μ produces an occupation measure $h = \mathcal{G}_{\mu,K} = \sum_{t\geq 0} \mu K^t$, and any family of sets that meets each K-trajectory at most r times automatically satisfies $\sum_{a\in A} h(a) \leq r\mu(V)$. We then show how such certificates arise systematically from a killed downward chain and a positive superharmonic weight. This abstracts the divisibility argument from the forum thread into a reusable framework. We also record a Boolean-lattice example recovering the classical Lubell–Yamamoto–Meshalkin inequality, and note that on finite acyclic digraphs the certificate-design problem becomes a literal linear flow problem.

We also record a brief reformulation of Erdős problem #858. Its forbidden relation is a transitive rough-divisibility order, so the problem fits the same certificate language; we keep that discussion short here, since it naturally deserves separate treatment.

The general theorems below are rigorous. The final sections contain applications, reformulations, and conjectural directions for several Erdős problems; those should be read as a research program, not as claimed resolutions.

1 Introduction

In the forum discussion of Erdős problem #1196, Liam Price summarized a new proof strategy based on a downward divisibility chain and its truncated adjoint. Tao then asked how much this idea could be reused elsewhere, and observed that while the idea looks suggested in hindsight by the earlier literature, it does not feel like a routine continuation of the published methods [4]. The purpose of this note is to package that idea into a general framework. For background on primitive sets beyond the forum thread, see Erdős' foundational paper and Lichtman's recent resolution of the Erdős primitive set conjecture [1, 18].

At a high level, the framework is a weighted infinite-poset analogue of the Lubell-Sperner random-chain method [13, 14, 16, 15]. Instead of sampling a maximal chain in a finite ranked poset, we build a sub-Markov process on a countable state space. The resulting occupation measure—a standard object from Markov-chain potential theory; see, for instance, [2]—plays the role of the Lubell weight. If the admissible sets in question can hit each sample path only a bounded number of times, then one gets an immediate weighted inequality. Proposition 2.9 below makes this analogy completely explicit on the Boolean lattice.

There are three levels to the framework.

1. A purely formal occupancy principle for sub-Markov kernels.

2. A general construction of such kernels from a *killed downward chain* and a positive superharmonic weight.

3. Problem-specific design questions: construct a certificate whose occupation measure dominates the weight of interest while keeping the total source mass small.

The last point is where the arithmetic begins. In the divisibility setting, the von Mangoldt kernel from the #1196 thread is one concrete instance. But the formalism itself is much broader and naturally suggests variants for other Erdős problems. One pleasant surprise is that the apparently threshold-dependent condition in problem #858 can also be recast in this language; we only record the basic reformulation here.

2 The general chain-certificate principle

Let V be a countable set.

Definition 2.1 (Sub-Markov kernel)**.** *A* sub-Markov kernel *on* V *is a function* $K : V \times V \to [0, \infty)$ *such that*

$$\sum_{y \in V} K(x, y) \leq 1 \qquad (x \in V).$$

Given a nonnegative measure μ *on* V*, we define the successive masses*

$$\mu_0 := \mu, \qquad \mu_{t+1}(y) := \sum_{x \in V} \mu_t(x) K(x, y) \qquad (t \geq 0),$$

and the occupation measure

$$\mathcal{G}_{\mu,K} := \sum_{t \geq 0} \mu_t,$$

whenever the series converges pointwise in $[0, \infty]$*. For certificates we will always require the source measure to have finite total mass, but allowing general nonnegative sources is convenient in the adjoint construction below.*

A sample path of K is a sequence $x_0, x_1, \ldots$ such that $K(x_i, x_{i+1}) > 0$ for each i until the process is killed.

Definition 2.2 (r-admissibility)**.** *Let* $r \in \mathbb{N}$*. A set* $A \subseteq V$ *is* r-admissible for K *if every sample path of* K *visits* A *at most* r *times.*

Theorem 2.3 (Weighted occupancy bound)**.** *Let* K *be a sub-Markov kernel on* V*, let* μ *be a finite nonnegative measure, and let* $h := \mathcal{G}_{\mu,K}$*. If* $A \subseteq V$ *is* r*-admissible for* K*, then*

$$\sum_{a \in A} h(a) \leq r\, \mu(V).$$

More generally, if $w : V \to [0, \infty)$ *satisfies* $w \leq h$ *pointwise, then*

$$\sum_{a \in A} w(a) \leq r\, \mu(V).$$

Proof. If $\mu(V) = 0$, there is nothing to prove. Otherwise start the chain from the initial law $\mu/\mu(V)$ and denote it by $(X_t)_{t\geq 0}$. Then

$$\sum_{a\in A} h(a) = \sum_{t\geq 0}\sum_{a\in A} \mu_t(a) = \mu(V)\,\mathbb{E}\Big[\sum_{t\geq 0} \mathbf{1}_A(X_t)\Big].$$

By r-admissibility the random sum inside the expectation is at most r, hence the claimed bound. The second statement follows immediately from $w \leq h$. □

Remark 2.4. *Theorem 2.3 is the core of the method. It may be viewed as an infinite weighted analogue of the classical random-chain proof of Sperner's theorem and the Lubell–Yamamoto–Meshalkin inequality; compare [14, 15, 16, 17].*

Definition 2.5 (Chain certificate)**.** *Let $\mathcal{F}$ be a family of subsets of V and let $w : V \to [0, \infty)$ be a target weight. An r-*chain certificate *for $(w, \mathcal{F})$ is a pair (μ, K) consisting of a finite measure μ and a sub-Markov kernel K on V such that*

- *(a) every $A \in \mathcal{F}$ is r-admissible for K;*
- *(b) the target weight is dominated by the occupation measure: $w \leq \mathcal{G}_{\mu,K}$.*

Its cost *is $r\mu(V)$.*

Corollary 2.6 (Certificate inequality)**.** *If (μ, K) is an r-chain certificate for $(w, \mathcal{F})$, then every $A \in \mathcal{F}$ satisfies*

$$\sum_{a\in A} w(a) \leq r\mu(V).$$

Consequently,

$$\sup_{A\in\mathcal{F}} \sum_{a\in A} w(a) \leq \inf_{(\mu,K)\ \textit{certificate}} r\mu(V).$$

Proposition 2.7 (Finite acyclic flow formulation)**.** *Let $G = (V, E)$ be a finite acyclic digraph, let $r \in \mathbb{N}$, and let $\mathcal{F}$ be a family of subsets of V such that every directed path of G meets each $A \in \mathcal{F}$ in at most r points. Let $w : V \to [0, \infty)$ be a target weight. Then the following are equivalent:*

- *(a) there exists an r-chain certificate (μ, K) for $(w, \mathcal{F})$ such that $K(u, v) = 0$ whenever $(u, v) \notin E$;*
- *(b) there exist nonnegative functions $h, \mu : V \to [0, \infty)$ and a nonnegative edge flow $F : E \to [0, \infty)$ such that*

$$\begin{aligned} w(v) &\leq h(v) \qquad (v \in V),\\ h(v) &= \mu(v) + \sum_{u:(u,v)\in E} F(u, v) \qquad (v \in V),\\ \sum_{v:(u,v)\in E} F(u, v) &\leq h(u) \qquad (u \in V). \end{aligned}$$

Under this correspondence, the certificate cost is $r\mu(V)$. In particular, minimizing the cost over all certificates supported on E is a finite linear program.

Proof. Assume (a), and set $h := \mathcal{G}_{\mu,K}$. Define $F(u,v) := h(u)K(u,v)$ on E. Since $h = \mu + hK$, for each $v \in V$ we have

$$h(v) = \mu(v) + \sum_{u \in V} h(u)K(u,v) = \mu(v) + \sum_{u:(u,v)\in E} F(u,v),$$

while

$$\sum_{v:(u,v)\in E} F(u,v) = h(u) \sum_{v \in V} K(u,v) \le h(u).$$

Thus (b) holds.

Conversely, assume (b). If $h(u) = 0$, the outgoing inequality forces $F(u,v) = 0$ for all $(u,v) \in E$. Hence we may define

$$K(u,v) := \begin{cases} F(u,v)/h(u), & h(u) > 0, \\ 0, & h(u) = 0. \end{cases}$$

Then K is a sub-Markov kernel supported on E, because

$$\sum_{v \in V} K(u,v) = \frac{1}{h(u)} \sum_{v:(u,v)\in E} F(u,v) \le 1$$

whenever $h(u) > 0$, and trivially when $h(u) = 0$. Moreover,

$$h(v) = \mu(v) + \sum_{u \in V} h(u)K(u,v) = \mu(v) + (hK)(v).$$

Because G is acyclic, there exists T with $K^T = 0$; iterating the identity $h = \mu + hK$ gives

$$h = \sum_{t=0}^{T-1} \mu K^t = \mathcal{G}_{\mu,K}.$$

Thus $w \le h = \mathcal{G}_{\mu,K}$, and Theorem 2.3 applies since every $A \in \mathcal{F}$ is r-admissible for any kernel supported on E. So (a) holds. ☐

Remark 2.8. *Proposition 2.7 turns certificate design on a finite acyclic graph into a concrete source-and-flow optimization problem, close in spirit to the flow-theoretic side of Sperner theory; compare Engel [17, Chapter 4].*

Proposition 2.9 (Boolean-lattice example: LYM and its m-chain extension)**.** *Let $V = 2^{[n]}$, let $\mu = \delta_\varnothing$, and define an upward kernel by*

$$K(S, S \cup \{i\}) = \frac{1}{n - |S|} \qquad (i \notin S),$$

with all other transition probabilities equal to 0. Then

$$\mathcal{G}_{\mu,K}(S) = \frac{1}{\binom{n}{|S|}} \qquad (S \subseteq [n]).$$

Consequently, every antichain $\mathcal{A} \subseteq 2^{[n]}$ satisfies

$$\sum_{S \in \mathcal{A}} \frac{1}{\binom{n}{|S|}} \le 1,$$

and more generally every family $\mathcal{A} \subseteq 2^{[n]}$ containing no chain of length $m+1$ satisfies

$$\sum_{S \in \mathcal{A}} \frac{1}{\binom{n}{|S|}} \le m.$$

Proof. A K-trajectory is precisely a uniformly random maximal chain

$$\varnothing = S_0 \subset S_1 \subset \cdots \subset S_n = [n],$$

obtained by adding one uniformly random new element at each step. Hence for each k, the state S_k is uniformly distributed among the k-subsets of $[n]$, so a fixed set S is visited with probability $\binom{n}{|S|}^{-1}$. Since the chain visits each state at most once, this is exactly the occupation measure.

An antichain meets every maximal chain at most once, so Theorem 2.3 gives the first inequality. If $\mathcal{A}$ contains no chain of length $m+1$, then every maximal chain meets $\mathcal{A}$ in at most m points, and the second inequality follows the same way. □

Remark 2.10. *Proposition 2.9 recovers the classical Lubell–Yamamoto–Meshalkin inequality and its standard m-chain extension inside the present framework; compare [14, 15, 16, 17].*

Proposition 2.11 (Lifted or augmented-state certificates)**.** *For each parameter θ in a countable index set Θ, let V_θ be a countable state space, let K_θ be a sub-Markov kernel on V_θ, let μ_θ be a finite measure on V_θ, and let $\pi_\theta : V_\theta \to U$ be a projection to a common base space U. Assume also that*

$$\sum_{\theta\in\Theta} \mu_\theta(V_\theta) < \infty.$$

Define

$$h_\theta := \mathcal{G}_{\mu_\theta, K_\theta}, \qquad H(u) := \sum_{\theta\in\Theta} \sum_{x\in\pi_\theta^{-1}(u)} h_\theta(x).$$

Suppose that for every $A \subseteq U$ in a family $\mathcal{F}$, every lifted set $\pi_\theta^{-1}(A)$ is r-admissible for K_θ. Then

$$\sum_{u\in A} H(u) \le r \sum_{\theta\in\Theta} \mu_\theta(V_\theta) \qquad (A \in \mathcal{F}).$$

Hence any weight $w \le H$ on U is bounded the same way.

Proof. Apply Theorem 2.3 on each fibre V_θ and sum over θ. □

Proposition 2.11 is useful when the forbidden relation depends on extra data not visible in the ambient poset. We include it here as a flexible formalism for future variants in which admissibility depends on hidden parameters or local thresholds.

3 Certificates from killed downward chains

We now describe a systematic way to manufacture chain certificates.

Definition 3.1 (Ranked downward kernel)**.** *A* ranked state space *is a countable set V equipped with a function $\rho : V \to \mathbb{N}$. A sub-Markov kernel P on V is* downward *if*

$$P(x, y) > 0 \implies \rho(y) < \rho(x).$$

A positive function $h : V \to (0, \infty)$ is P-superharmonic *if*

$$\sum_{x\in V} h(x)P(x, y) \le h(y) \qquad (y \in V).$$

The idea is to reverse the downward kernel with respect to the weight h.

Theorem 3.2 (Adjoint-certificate construction)**.** *Let P be a downward sub-Markov kernel on a ranked state space V, and let $h : V \to (0,\infty)$ be P-superharmonic. Define the upward adjoint kernel K_h by*

$$K_h(u,v) := \frac{h(v)}{h(u)}P(v,u) \qquad (u,v \in V),$$

with the convention that $P(v,u) = 0$ unless $\rho(u) < \rho(v)$. Then:

(i) K_h is a sub-Markov kernel on V;

(ii) if

$$\mu_h(u) := h(u)\Big(1 - \sum_{v\in V} P(u,v)\Big),$$

then h is exactly the occupation measure of (μ_h, K_h), i.e.

$$h = \mathcal{G}_{\mu_h,K_h};$$

(iii) if in addition $\mu_h(V) < \infty$, then any r-admissible set $A \subseteq V$ for K_h satisfies

$$\sum_{a\in A} h(a) \le r \sum_{u\in V} \mu_h(u).$$

Proof. For each $u \in V$,

$$\sum_{v\in V} K_h(u,v) = \frac{1}{h(u)} \sum_{v\in V} h(v)P(v,u) \le 1$$

by P-superharmonicity, so K_h is sub-Markov.

Next, for each fixed $v \in V$ we compute

$$\begin{aligned}
\mu_h(v) + \sum_{u\in V} h(u)K_h(u,v) &= h(v)\Big(1 - \sum_{w\in V} P(v,w)\Big) + \sum_{u\in V} h(u)\frac{h(v)}{h(u)}P(v,u) \\
&= h(v)\Big(1 - \sum_{w\in V} P(v,w)\Big) + h(v)\sum_{u\in V} P(v,u) \\
&= h(v).
\end{aligned}$$

Thus $h = \mu_h + hK_h$.

Because K_h is upward with respect to the ranking, each vertex of rank at most R only depends on vertices of smaller rank in the identity $h = \mu_h + hK_h$. An induction on the rank therefore gives

$$h = \sum_{t\ge 0} \mu_h K_h^t = \mathcal{G}_{\mu_h,K_h}.$$

If $\mu_h(V) < \infty$, the last claim is now Theorem 2.3. □

Remark 3.3. *Theorem 3.2 isolates the exact algebra behind the #1196 argument. In applications one must additionally verify that the source mass $\mu_h(V)$ is finite. The arithmetic work is then split into two tasks:*

1. *choose a downward killed chain P for which admissible sets are sparse along trajectories;*
2. *find a positive superharmonic weight h that is as close as possible to the target weight.*

The total source mass $\sum \mu_h$ *is then the cost of the certificate.*

Corollary 3.4 (m-chain families)**.** *In the setting of Theorem 3.2, assume* $\mu_h(V) < \infty$*. If every directed path of the upward adjoint* K_h *meets a set* A *in at most* m *points, then*

$$\sum_{a \in A} h(a) \le m \sum_{u \in V} \mu_h(u).$$

This simple m-hitting variant is already enough to suggest new weighted inequalities beyond the primitive-set case.

4 The divisibility certificate behind problem #1196

Let $x \ge 2$ and $Y \ge 2$. On the state space

$$V_x := \{n \in \mathbb{N} : n \ge x\}$$

consider the downward kernel

$$P_{x,Y}(n,m) := \begin{cases} \dfrac{\Lambda(n/m)}{\log n}, & m \mid n,\ m < n,\ n/m \ge Y,\ m \ge x, \\ 0, & \text{otherwise}. \end{cases}$$

This is sub-Markov because $\sum_{q|n} \Lambda(q) = \log n$ and we have only kept some of the divisors.

Set

$$h_x(n) := \frac{1}{n \log n} \qquad (n \ge x).$$

Proposition 4.1 (Divisibility adjoint certificate)**.** *Assume that*

$$\sum_{q \ge Y} \frac{\log n}{\log^2(nq)} \frac{\Lambda(q)}{q} \le 1 \qquad (n \ge x), \tag{1}$$

where the sum runs over prime powers q*. Then* h_x *is* $P_{x,Y}$*-superharmonic, and the upward adjoint kernel is*

$$K_{x,Y}(n, nq) = \frac{\log n}{\log^2(nq)} \frac{\Lambda(q)}{q} \qquad (n \ge x,\ q \ge Y).$$

The corresponding source measure is

$$\mu_{x,Y}(n) = \frac{1}{n \log n} \left(1 - \frac{1}{\log n} \sum_{\substack{q|n \\ q \ge Y,\ n/q \ge x}} \Lambda(q) \right) \qquad (n \ge x),$$

and

$$\frac{1}{n \log n} = \mathcal{G}_{\mu_{x,Y}, K_{x,Y}}(n) \qquad (n \ge x).$$

Proof. For any $n \ge x$,

$$\begin{aligned} \sum_{m \in V_x} h_x(m) P_{x,Y}(m,n) &= \sum_{q \ge Y} \frac{1}{nq \log(nq)} \frac{\Lambda(q)}{\log(nq)} \\ &= \frac{1}{n \log n} \sum_{q \ge Y} \frac{\log n}{\log^2(nq)} \frac{\Lambda(q)}{q} \\ &\le \frac{1}{n \log n} = h_x(n) \end{aligned}$$

by (1). Thus h_x is superharmonic, and Theorem 3.2 gives the formula for $K_{x,Y}$ and $\mu_{x,Y}$. ☐

Every $K_{x,Y}$-trajectory is a divisibility chain

$$n_0 \mid n_1 \mid n_2 \mid \cdots, \qquad n_{j+1} = n_j q_j,$$

so any family that is sparse on divisibility chains is automatically admissible.

Definition 4.2. *A set $A \subseteq \mathbb{N}$ is m-*sparse on divisibility chains *if every infinite divisibility chain meets A in at most m points.*

Remark 4.3. *The case $m = 1$ is exactly the classical notion of a primitive set, going back to Erdős [1].*

Corollary 4.4 (Generalized tail inequality)**.** *Let $A \subseteq [x, \infty) \cap \mathbb{N}$ be m-sparse on divisibility chains. Under the hypothesis (1),*

$$\sum_{a \in A} \frac{1}{a \log a} \leq m\, B_{x,Y}, \qquad B_{x,Y} := \sum_{n \geq x} \mu_{x,Y}(n).$$

In particular, every primitive set satisfies the bound with $m = 1$.

Proof. Apply Corollary 3.4 to the certificate from Proposition 4.1. □

Remark 4.5. *The forum-thread argument shows, by standard Mertens-type estimates, that for every fixed large Y one has*

$$B_{x,Y} = 1 + O_Y\left(\frac{1}{\log x}\right) \qquad (x \to \infty).$$

Thus Corollary 4.4 packages the claimed #1196 result when $m = 1$.

The same certificate immediately yields a new weighted corollary for one other Erdős problem.

Lemma 4.6. *If $A \subseteq \mathbb{N}$ has no three distinct elements $a, b, c \in A$ with $a \mid b$ and $a \mid c$, then A is 2-sparse on divisibility chains.*

Proof. If a divisibility chain contained three elements $a \mid b \mid c$ of A, then $a \mid b$ and $a \mid c$ with a, b, c distinct, contrary to the assumption. □

Corollary 4.7 (A chain-certificate consequence for problem #1062)**.** *Assume (1). If $A \subseteq [x, \infty) \cap \mathbb{N}$ has no three distinct elements a, b, c with $a \mid b$ and $a \mid c$, then*

$$\sum_{a \in A} \frac{1}{a \log a} \leq 2 B_{x,Y}.$$

Hence, using the Mertens evaluation of $B_{x,Y}$ from the preceding remark,

$$\sum_{a \in A} \frac{1}{a \log a} \leq 2 + o(1) \qquad (x \to \infty).$$

Proof. Combine Lemma 4.6 with Corollary 4.4 for $m = 2$. □

So the same kernel that treats primitive sets also gives an immediate factor-2 extension to families arising in problem #1062.

5 A dictionary of certificate-design problems

This section records several Erdős problems as chain-certificate design problems. Some of these reformulations are rigorous and direct; others are more speculative. The point is to make the reusable structure explicit.

1. Problem #1196

For problem #1196 [3], let

$$w_x(n) := \frac{\mathbf{1}_{n\geq x}}{n\log n},$$

and let $\mathcal{P}_x$ be the family of primitive subsets of $[x,\infty)\cap\mathbb{N}$. The forum construction produces, for each fixed sufficiently large Y and all sufficiently large x, a 1-certificate for $(w_x,\mathcal{P}_x)$ of cost $B_{x,Y} = 1 + O_Y(1/\log x)$. Thus the problem is naturally a certificate problem at cost $1+o(1)$, matching the $1+o(1)$ scale now recorded for the problem [4].

2. Problem #164

For problem #164 [5], the family is all primitive subsets of $\{2,3,\ldots\}$, and the target weight is

$$w(n) := \frac{1}{n\log n}.$$

The problem is already solved by other means, for instance by Lichtman [18], but the certificate viewpoint suggests an alternate goal: build a global 1-certificate whose cost is exactly

$$\sum_p \frac{1}{p\log p},$$

perhaps with source measure concentrated on primes, or at least identify a dual variational principle whose extremiser is the set of primes. This is a reformulation rather than a new proof, but it clarifies what an occupation-measure proof would need to accomplish.

3. The stronger speculation inside problem #542

Problem #542 [6] asks about families $A\subseteq\{1,\ldots,N\}$ with $[a,b]>N$ for all distinct $a,b\in A$. Such a family is automatically primitive, since $a\mid b$ would imply $[a,b]=b\leq N$. Erdős is recorded on the problem page as further speculating that

$$\sum_{a\in A}\frac{1}{a}\leq 1+o(1).$$

From the certificate viewpoint, this becomes:

Question 5.1 (Certificate reformulation of the #542 sharpening)**.** *Can one construct a* 1*-certificate for the family of* $[a,b]>N$ *sets with target weight comparable to* $\mathbf{1}_{n\leq N}/n$ *and total cost* $1+o(1)$*?*

Because the family is primitive, any such certificate would automatically be compatible with divisibility chains. The real difficulty is not admissibility but producing occupation weight of size $1/n$ instead of $1/(n\log n)$.

4. Problem #1062

For problem #1062 [12], let $\mathcal{F}_N$ be the family of subsets $A \subseteq \{1, \dots, N\}$ with no three distinct elements a, b, c satisfying $a \mid b$ and $a \mid c$. By Lemma 4.6, every such set is 2-sparse on divisibility chains. Therefore the remaining challenge is to find a certificate whose occupation measure is *close to uniform* on $\{1, \dots, N\}$.

Question 5.2 (Uniform 2-certificate problem)**.** *Is there a* 2*-certificate for* $\mathcal{F}_N$ *whose occupation measure dominates a constant multiple of* $\mathbf{1}_{[1,N]}$ *and whose cost is asymptotic to the true value of* $f(N)$*?*

Any such certificate would translate immediately into upper bounds for $f(N)$. The point is that the combinatorics of #1062 already gives the factor 2; what remains is a measure-design problem.

5. Problem #856

The page for #856 [7] notes a one-step counting argument: for every t, there are fewer than k representations $t = ap$ with $a \in A$ and p prime, which yields

$$\sum_{a \in A} \frac{1}{a} \sum_{p<N} \frac{1}{p} \ll k \sum_{t<N^2} \frac{1}{t}.$$

This is already the shadow of a certificate. One may regard it as a depth-1 prime-extension process.

The natural next question is whether a *multi-step* prime-extension kernel can be built so that admissible sets for #856 have bounded occupancy along sample paths. Any positive answer would turn the current one-step bound into a genuine chain-certificate argument and might sharpen the known estimate $f_k(N) \ll \log N / \log\log N$.

6. Problem #858

For problem #858 [8], define a strict relation on $\mathbb{N}$ by

$$a \prec_{\mathrm{r}} b \quad \Longleftrightarrow \quad b = at \text{ for some integer } t \geq 2 \text{ with } P^-(t) > a.$$

This relation is transitive: if $b = at_1$ with $P^-(t_1) > a$ and $c = bt_2$ with $P^-(t_2) > b > a$, then $c = a(t_1 t_2)$ and every prime factor of $t_1 t_2$ still exceeds a. Hence the admissible sets in problem #858 are exactly the antichains of the rough-divisibility order $(\{1, \dots, N\}, \prec_{\mathrm{r}})$.

Therefore any sub-Markov kernel K on $\{1, \dots, N\}$ supported on $\prec_{\mathrm{r}}$, together with any finite source measure ν, yields a certificate by Theorem 2.3: every admissible A for #858 satisfies

$$\sum_{n \in A} w(n) \leq \nu(\{1, \dots, N\})$$

for every target weight $w \leq \mathcal{G}_{\nu,K}$. A natural first model is to let $K(a, at)$ be proportional to $1/t$ over the rough factors t with $P^-(t) > a$. The remaining task is then purely a design question: choose K and ν so that the occupation measure dominates $1/(n \log N)$ as uniformly as possible while keeping the total source mass as small as possible.

7. Problem #143

Problem #143 [11] replaces exact divisibility by the approximate condition

$$|kx - y| \geq 1 \qquad (x \neq y,\ k \in \mathbb{N}).$$

For integer sets this implies primitiveness, but the real-valued setting is no longer a literal poset. The certificate philosophy suggests an approximate version:

Question 5.3 (Fuzzy certificate problem for #143)**.** *Can one discretise the multiplicative half-line into shells and build a random multiplicative chain for which every set satisfying the #143 condition has bounded path occupancy up to controlled overlap?*

A positive answer would provide a chain-based route to the sparsity conclusions sought in #143. At present this is only a programmatic reformulation.

8. Problem #892

The necessary conditions recorded on the #892 page [10],

$$\sum_n \frac{1}{b_n \log b_n} < \infty \qquad \text{and} \qquad \sum_{b_n < x} \frac{1}{b_n} = o\left(\frac{\log x}{\sqrt{\log \log x}}\right),$$

are exactly the kind of obstructions that chain certificates produce. Thus #892 can be read as asking for a partial converse to the certificate method: when do such capacity obstructions already guarantee the existence of a primitive sequence with the prescribed scale?

9. Problem #872

The saturation game in #872 [9] lives on the same divisibility poset as primitive sets. A speculative possibility is to use certificate mass of the remaining legal region as a potential function. This is less developed than the other reformulations, but it is another place where the chain language may be useful.

6 Concluding remarks

The main lesson is that the #1196 proof idea is not an isolated trick. It belongs to a general mechanism:

> choose a downward killed chain, find a positive superharmonic weight, reverse it, and use bounded path occupancy to obtain a weighted inequality.

The arithmetic difficulty is concentrated in the superharmonic estimate and in keeping the source mass small.

From this viewpoint, several Erdős problems become instances of a single meta-problem:

Design a low-cost chain certificate whose occupation measure dominates the weight of interest.

On finite acyclic graphs, Proposition 2.7 shows that this is literally a flow optimization problem. For primitive sets, the divisibility chain is the natural geometry. For weakened or approximate notions of primitiveness, one should expect augmented state spaces, lifted chains, or fuzzy certificates.

I suspect the most promising next targets for this framework are:

1. the sharper $1 + o(1)$ speculation attached to problem #542;
2. the factor-2 world around problem #1062;
3. quantitative domination problems for the rough-divisibility order of problem #858.

The first asks for a better occupation weight, the second for a more uniform one, and the third for sharper control of occupation measures on a new transitive DAG.

References

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