A COVOLUME CRITERION FOR GRH

Abstract. We give a compact criterion for the generalized Riemann hypothesis for Dirichlet L-functions based on a comparison of squared covolumes. Its arithmetic hypotheses are short integral forms, almost-full spanning by a small collection of translates, and a matching determinant lower bound. We prove the scalar and index assembly, the finite-dimensional spanning argument, the determinant comparison, and the passage from the resulting prime-sum inequality to a zero-free half-plane. The organization follows the two editions of the *grh-lean* formal development, with classical results separated from estimates specific to its arithmetic construction. The latter estimates remain explicit hypotheses of this paper; an unconditional application requires their proofs in addition to the reductions given here.

1. Introduction

The argument considered here compares the same arithmetic volume in two ways. Small integral vectors give an upper bound. A product formula, applied to a family of twisted determinants, gives a lower bound. The leading terms agree. Their cancellation bounds a weighted sum of nonsplit primes, and a one-sided analytic criterion then excludes zeros of Dirichlet L-functions to the right of the critical line.

This organization is particularly clear in the compact edition of [4]. Its three central propositions are called `Data.short_forms`, `Data.enough_forms`, and `Data.integral_forms_long`. Their roles are different: the first constructs vectors with controlled denominators and norms; the second shows that a logarithmic number of translates supplies almost all directions; the third prevents such an almost spanning integral family from having too small a covolume. Keeping these roles distinct makes the numerical argument short and exposes the exact arithmetic work on which it depends.

The scope of the present paper is an explicit reduction theorem. The local repair of divisibility, the sharp second moment for the Poincaré map, and the congruence bounds for a prime-dependent family of twists are supplied as arithmetic hypotheses. We prove their stated consequences, including the final implication to GRH, and explain where the formal source establishes the hypotheses. Thus every theorem proved here has its dependencies in its statement. This manuscript does not assert that those three substantial arithmetic constructions have been independently reconstructed or verified in prose.

The resulting criterion can be stated without introducing the full machinery of unitary groups. The notation in the statement is defined in Sections 2 and 3. Its point is the scale of the error, not an optimized constant.

Theorem 1.1 (Covolume criterion)**.** *Suppose that the compatible arithmetic data of Hypothesis 3.1 exist for every modulus M divisible by* 18480*, every prime $d \equiv M/2 - 1 \pmod{M}$, every $0 < \varepsilon < 1/100$, and every sufficiently large real x (with the threshold allowed to depend on M, d, ε). Then, for every such fixed M, d and every $\eta > 0$,*

$$R_{d,M}(x) \le x^{-1/2+\eta} \qquad \text{for all sufficiently large } x. \tag{1.1}$$

Consequently, for every Dirichlet character χ, $L(s,\chi)$ has no zero with $\operatorname{Re} s > 1/2$, the pole at $s = 1$ for a principal character being excepted. Every zero in $0 < \operatorname{Re} s < 1$ therefore has real part $1/2$.

The classical background consists of Dirichlet L-functions, the prime number theorem in a fixed progression, Landau's theorem for integrals of one sign, character mean values, and Hermitian linear algebra. The source's division between background and new material is a working classification. A citation to a general theta or congruence theorem suffices only after its integral structures, normalizations, and uniform error terms have been matched to the assertion

needed here. The construction-specific content is concentrated in the three arithmetic estimates and their compatibility for one set of data.

The quantitative part of the proof is the cancellation

$$D(H-3L) \;\le\; V+2\delta \;\le\; n(H-m\mathcal{R}+L)+2\delta, \tag{1.2}$$

where D is the row dimension, $n \ge (1-4b^{-2})D$, and $m \asymp \sqrt{x}$. The index cost is at most $5D\log(1+b^2)\mathcal{B}$ and $L = \mathcal{B}\log b$. The missing directions cost only $4Dx^{99}$. After cancelling DH, one obtains $m\mathcal{R} \le fa(\log b)^5$. Neither an estimate of H at its natural scale nor a filtration of the row space is needed at this point. The exponent $1/2$ comes from the length m of the chain of lifts.

Only the principal propositions retain source names. Auxiliary arguments are stated in conventional mathematical language. An appendix records the corresponding source files and the status of the arithmetic inputs. The source snapshot used throughout is commit `72fead102f663d5c7a407964a13dc0cf9d977207` of [4]; its documentation attributes the underlying proof to Levent Alpöge, with Claude, and the shorter rewriting to Sernau and colleagues at Anthropic. That attribution is distinct from a claim of priority for the elementary reorganizations made here.

2. The prime-sum error

Let $M \ge 1$ and let d be an odd prime prime to M. Write $\chi_d(n) = (n/d)$ for the quadratic character, Λ for the von Mangoldt function, and $c_M = 1/\varphi(M)$. Characters are extended by zero away from the units of their modulus. Define

$$\lambda_{d,M}(n) = \Lambda(n)(1-\chi_d(n))\mathbf{1}_{n\equiv 1 \pmod M}. \tag{2.1}$$

This coefficient is nonnegative. For $d \equiv 7 \pmod 8$, it counts prime powers associated with the imaginary quadratic field $\mathbb{Q}(\sqrt{-d})$, with the weights in (2.1); the prime d itself is retained with its prescribed weight. There is no omission of ramified primes or prime powers in the formulas that follow.

For a character ξ, let $\widetilde{L}(s,\xi) = L(s,\xi)$ when ξ is nonprincipal, and let $\widetilde{L}(s,\xi) = (s-1)L(s,\xi)$ when it is principal, with the removable value taken at 1. These are entire functions. Put

$$B_{d,M}(s) = c_M \left\{ \sum_{\xi \bmod M} \left(\frac{\widetilde{L}'}{\widetilde{L}}(s,\xi\chi_d) - \frac{\widetilde{L}'}{\widetilde{L}}(s,\xi) \right) - 1 \right\}. \tag{2.2}$$

Here $\xi\chi_d$ is a character modulo Md. It is never principal: by the Chinese remainder theorem one can prescribe $n \equiv 1 \pmod M$ and $\chi_d(n) = -1$. The classical nonvanishing theorem on $\operatorname{Re} s = 1$ shows that $B_{d,M}$ is holomorphic near 1. Character conjugation shows that its value there is real. Set $\kappa_{d,M} = B_{d,M}(1)$ and, for real $x \ge 1$, define

$$R_{d,M}(x) = \sum_{n\le x} \lambda_{d,M}(n)\left(\frac{1}{n} - \frac{1}{x}\right) - c_M \log x - \kappa_{d,M}. \tag{2.3}$$

The subtraction of the constant is essential. An arbitrary additive constant would introduce a pole at 1 into the transform used later. For $M = 1$, the Laurent expansion of ζ gives $\kappa_{d,1} = -1-\gamma+L'(1,\chi_d)/L(1,\chi_d)$, where γ is Euler's constant. Thus if $\ell_d = \gamma - L'(1,\chi_d)/L(1,\chi_d)$, then $\kappa_{d,1} = -1-\ell_d$.

Lemma 2.1 (Dirichlet series and Mellin transform)**.** *For* $\operatorname{Re} s > 1$*,*

$$\sum_{n\ge 1} \frac{\lambda_{d,M}(n)}{n^s} = \frac{c_M}{s-1} + B_{d,M}(s) + c_M. \tag{2.4}$$

The transform $\Phi_{d,M}(s) = \int_1^\infty R_{d,M}(x)x^{-s}\,dx$*, initially defined for* $\operatorname{Re} s > 2$*, has the meromorphic continuation*

$$\Phi_{d,M}(s) = \frac{B_{d,M}(s)/s - \kappa_{d,M}}{s-1}. \tag{2.5}$$

It is holomorphic on $\operatorname{Re} s \geq 1$. *In* $\operatorname{Re}\rho > 0$ *its poles are at most simple, with residue*

$$\operatorname*{Res}_{s=\rho} \Phi_{d,M}(s) = \frac{\sum_{\xi \bmod M} \big(m_\xi(\rho) - m_{\xi\chi_d}(\rho)\big)}{\varphi(M)\rho(1-\rho)}, \tag{2.6}$$

where $m_\xi(\rho)$ *is the order of vanishing of* $\widetilde{L}(s,\xi)$ *at* ρ, *and the residue at the removable point* 1 *is zero.*

Proof. Orthogonality of characters gives $\mathbf{1}_{n\equiv 1 \pmod M} = c_M \sum_\xi \xi(n)$. The Euler product yields $-L'/L(s,\xi) = \sum_n \Lambda(n)\xi(n)n^{-s}$. Replacing the principal L-function by $(s-1)L(s,\xi)$ contributes $c_M/(s-1)$, proving (2.4). Absolute convergence justifies each interchange.

For $\operatorname{Re} s > 2$, integration term by term gives

$$\int_1^\infty \sum_{n\leq x} \lambda_{d,M}(n)(n^{-1}-x^{-1})x^{-s}\,dx = \frac{1}{s(s-1)} \sum_{n\geq 1} \frac{\lambda_{d,M}(n)}{n^s}.$$

The transforms of $\log x$ and 1 are $(s-1)^{-2}$ and $(s-1)^{-1}$. Substituting (2.4) and combining the three rational terms gives (2.5). The apparent pole at 1 is removable because $B_{d,M}(1) = \kappa_{d,M}$. At any other point of the right half-plane, the logarithmic derivative of a nonzero entire function has a simple pole with residue equal to its order of vanishing. This proves (2.6), including cancellation when the numerator vanishes. □

We use the prime number theorem only with M and d fixed. This matters: a uniform zero-free region for growing moduli is not being assumed. In particular, a possible exceptional real zero for one fixed modulus merely changes the constants and the threshold.

Lemma 2.2 (The error at a distant cutoff)**.** *For every fixed* M, d *and every* $A > 0$,

$$R_{d,M}(x) = O_{M,d,A}\big((\log x)^{-A}\big). \tag{2.7}$$

In fact the classical fixed-modulus prime number theorem gives $R_{d,M}(x) = O_{M,d}(\exp(-c\sqrt{\log x}))$ *for some* $c > 0$.

Proof. The prime number theorem in progressions, applied modulo Md, gives $A_0(t) := \sum_{n\leq t} \lambda_{d,M}(n) = c_M t + E(t)$ with $E(t) = O_{M,d}(t\exp(-c_0\sqrt{\log t}))$; see [3, Chapter 11]. This also follows by summing over the residue classes in (2.1); powers of the finitely many primes dividing Md contribute $O_{M,d}(\log t)$ and are included in the error. Partial summation gives

$$\sum_{n\leq x} \lambda_{d,M}(n)(n^{-1}-x^{-1}) = \int_1^x \frac{A_0(t)}{t^2}\,dt = c_M \log x + \int_1^x \frac{E(t)}{t^2}\,dt.$$

The constant $\int_1^\infty E(t)t^{-2}\,dt$ equals $\kappa_{d,M}$. Indeed, partial summation in the Dirichlet series gives $\sum_n \lambda_{d,M}(n)n^{-s} = c_M s/(s-1) + s\int_1^\infty E(t)t^{-s-1}\,dt$; comparison with (2.4) and passage to $s=1$ identifies the constant. Hence $R_{d,M}(x) = -\int_x^\infty E(t)t^{-2}\,dt$. Substituting $u = \sqrt{\log t}$ bounds the tail by a constant times $(1+\sqrt{\log x})e^{-c_0\sqrt{\log x}}$, which gives both assertions after decreasing c_0. □

The arithmetic construction uses two cutoffs $x < y$ and a long order $a \geq y$. The next identity explains why its remainder is precisely (2.3), rather than an asymptotically equivalent prime count.

Lemma 2.3 (Two weights)**.** *Let* $\mathcal{T}$ *be a set of* T *indices, let* $0 < S \leq T$, *and suppose* $f = \varphi(M)T$. *For each* $\nu \leq a$ *assign to each* $\tau \in \mathcal{T}$ *a cutoff* $z_{\nu,\tau} \in \{x, y\}$, *with exactly* $S\mathbf{1}_{\nu\equiv 1 \pmod M}$ *of these cutoffs equal to* x. *Put*

$$J_\tau = \sum_{\nu\leq a} \lambda_{d,1}(\nu)\left(\frac{a}{\nu} - \frac{a}{z_{\nu,\tau}}\right)_+, \qquad \overline{w} = \log a - \frac{S}{f}\log x - \left(1-\frac{S}{f}\right)\log y,$$

where $u_+ = \max(u,0)$, *and set* $\mathcal{R} = \sum_{\tau\in\mathcal{T}} J_\tau - Ta(\log a - 1 - \ell_d - \overline{w})$. *Then*

$$\mathcal{R} = aS\big(R_{d,M}(x) - R_{d,M}(y)\big) + aTR_{d,1}(y). \tag{2.8}$$

Proof. Write $P_M(z) = \sum_{\nu \le z} \lambda_{d,M}(\nu)(\nu^{-1} - z^{-1})$. Since $z \le a$, extending the sum to $\nu \le a$ and taking the positive part leaves its value unchanged. Counting the cutoffs equal to x therefore gives $\sum_\tau J_\tau = aS(P_M(x) - P_M(y)) + aTP_1(y)$. Substitute $P_M(z) = R_{d,M}(z) + c_M \log z + \kappa_{d,M}$. The constants for modulus M cancel in the difference, and the remaining main terms are $aSc_M(\log x - \log y) + aT(\log y - 1 - \ell_d)$, exactly the subtracted expression because $f = \varphi(M)T$. □

In the source, $\mathcal{T}$ is the set of top infinite places and S is the number of thick places. Their orbit-counting property gives the cutoff count in the lemma. Constructing such a set with adequate size is part of the arithmetic data; the identity itself is the finite calculation above.

3. The covolume comparison

We first state the exact interface between the arithmetic construction and the numerical argument. A complex vector space W of dimension D is the row space. A space V of dimension C is the full column space, and $\widehat{V} \subseteq V$ is its ramified subspace. There are maps $B_s : \widehat{V} \to W$ indexed by a finite set Γ. A subset $\mathcal{L} \subseteq \widehat{V}$ consists of integral columns and spans $\widehat{V}$. The word integral here refers to specified lattices at every finite place, not to the choice of a complex basis.

There is a family of injective lifts $\Theta_\vartheta : W \to H_\vartheta$, one for each allowed twist, including the trivial twist 1. In the arithmetic realization, the allowed twists are the tame characters whose first m positive powers remain nontrivial at their support, as specified in Section 4. The target spaces have positive Hermitian forms at the complex realizations σ. For a finite row family $\mathbf{y} = (y_1, \dots, y_n)$ write

$$\Delta_\sigma(\mathbf{y}) = \det\big(\langle \Theta_1 y_i, \Theta_1 y_j \rangle_\sigma\big)_{i,j=1}^n. \tag{3.1}$$

This is squared covolume. In particular, Hadamard's inequality gives $\Delta_\sigma(\mathbf{y}) \le \prod_i \|\Theta_1 y_i\|_\sigma^2$. Statements quantified over σ refer to the arithmetic realizations of the source and their corresponding Hermitian forms. Applying an arbitrary automorphism of $\mathbb{C}$ to the entries of a Gram matrix need not preserve positivity; that issue is addressed explicitly in Section 6.

An index bound has an equally specific meaning. For every rational prime p and every field embedding $\iota : \mathbb{C} \to \mathbb{C}_p$, with no continuity requirement, let $\mathcal{O}_\iota$ denote the transported valuation ring and let $\mathcal{I}_{\vartheta,\iota} \subset H_\vartheta$ be the prescribed integral module. Here $\mathbb{C}_p$ is the completion of an algebraic closure of $\mathbb{Q}_p$. We say that $\mathbf{y}$ has index at most δ, and write $\mathrm{ind}(\mathbf{y}) \le \delta$, if there are nonnegative numbers e_p with $\sum_p e_p \le \delta$ such that, whenever the twist is unramified at p, there is a matrix $U \in M_n(\mathcal{O}_\iota)$ satisfying

$$|\det U|_\iota \ge e^{-e_p}, \qquad \Theta_\vartheta\left(\sum_j U_{ij} y_j\right) \in \mathcal{I}_{\vartheta,\iota} \quad (1 \le i \le n). \tag{3.2}$$

The sum over primes means the supremum of finite subsums. The matrix may depend on p, ι, ϑ. This is the quantifier order in the compact source. The same budgets e_p must work for all these choices; a stronger requirement that the same matrix work for every twist is unnecessary for the numerical argument.

Arithmetic hypothesis 3.1 (Compatible arithmetic data)**.** Fix M, d as in Theorem 1.1 and $0 < \varepsilon < 1/100$. For every sufficiently large real x, suppose the preceding spaces, maps, lattices, and lifts are given with positive integers a, f, m, S, T, real numbers $b, y > 1$, and a real number H, where $D, C > 0$, and

$$\log b = x^\varepsilon, \quad \log y = x^{\varepsilon/6}, \quad \sqrt{x} \le m \le 2\sqrt{x}, \quad a \ge xb, \quad f = \varphi(M)T, \quad 1 \le T/S \le \log b. \tag{3.3}$$

Put $\mathcal{B} = fa(\log b)^3$, $L = \mathcal{B} \log b$, define $\mathcal{R}$ by (2.8), and set $h = H - m\mathcal{R} + L$. Assume the crude bound

$$|H| + m|\mathcal{R}| \le b^2 x^{99}. \tag{3.4}$$

Assume also the following three compatible estimates.

(A) Short forms. There is a spanning subset $\mathcal{S} \subseteq \mathcal{L}$ such that $\|\Theta_1 B_s u\|_\sigma^2 \le e^h$ for every $u \in \mathcal{S}$, every $s \in \Gamma$, and every σ. Every finite row family chosen from $\bigcup_{s\in\mathcal{U}} B_s(\mathcal{L})$, for $\mathcal{U} \subseteq \Gamma$, has index at most $|\mathcal{U}|C\mathcal{B}$.

(B) Enough forms. There is $\mathcal{U} \subseteq \Gamma$ such that

$$|\mathcal{U}|C \le 5D\log(1+b^2), \qquad \dim \sum_{s\in\mathcal{U}} B_s(\widehat{V}) \ge (1-4b^{-2})D. \tag{3.5}$$

(C) Integral forms are long. If $\mathbf{y}$ is a linearly independent family of $n \ge (1-4b^{-2})D$ rows, $\operatorname{ind}(\mathbf{y}) \le \delta$, and $\Delta_\sigma(\mathbf{y}) \le e^V$ for every σ, then

$$D(H-3L) \le V + 2\delta. \tag{3.6}$$

The hypothesis asks for one compatible construction. Separate examples satisfying (A), (B), and (C) with different levels, lattices, periods, or row spaces would not suffice. The rational scaling used in (A) does not alter the image subspaces in (B). The same scaling must, however, be used in the integral index statement and in the archimedean norm bound.

The source has further admissibility conditions, including $(\log b)^{100} \le x$, $x^{10^6} \le y$, $y^{10^6} \le b$, and $(\log y)^6 \le \log b$. The choices in (3.3) satisfy these for large x when $0 < \varepsilon < 1/100$. The remaining levels, ranks, and finite sets are needed to prove (A)–(C), but do not enter the argument of this section except through those estimates. The source's existence theorem and these further arithmetic requirements are recorded in the appendix, without treating their existence as a consequence of the displayed scale inequalities alone.

Proposition 3.2 (`Data.exists_family_covolumeSq_le`)**.** *Under* (A) *and* (B) *there is a linearly independent family* $\mathbf{y}$ *of* $n \ge (1-4b^{-2})D$ *rows with*

$$\operatorname{ind}(\mathbf{y}) \le \delta := 5D\log(1+b^2)\mathcal{B}, \qquad \Delta_\sigma(\mathbf{y}) \le e^{nh} \quad \text{for all } \sigma. \tag{3.7}$$

Proof. Choose $\mathcal{U}$ as in (B). Because $\mathcal{S}$ spans $\widehat{V}$, the set $\bigcup_{s\in\mathcal{U}} B_s(\mathcal{S})$ spans $\sum_{s\in\mathcal{U}} B_s(\widehat{V})$. Select a basis from this set. Its cardinality has the required lower bound. The index assertion in (A) applies to the selected family and gives the first inequality in (3.7); the second follows from Hadamard's inequality and the bound on each vector. No claim that the selected vectors form an integral basis is required. □

Proposition 3.3 (`star_inequality`)**.** *Under Hypothesis 3.1, for all sufficiently large* x*,*

$$m\mathcal{R} \le (\log b)^2\mathcal{B} = fa(\log b)^5. \tag{3.8}$$

Proof. Use the family of Proposition 3.2. Independence gives $n \le D$. Also $-h \le |H| + m|\mathcal{R}| \le b^2x^{99}$, because $L > 0$. Thus (C) gives the single comparison

$$\begin{aligned} D(H-3L) &\le nh + 2\delta = Dh + (D-n)(-h) + 2\delta \\ &\le D(H - m\mathcal{R} + L + 4x^{99} + 10\log(1+b^2)\mathcal{B}). \end{aligned}$$

Here $D - n \ge 0$ justifies using the upper bound for $-h$; no sign assumption on h is made. Divide by $D > 0$ and cancel H. The result is

$$m\mathcal{R} \le 4L + 4x^{99} + 10\log(1+b^2)\mathcal{B}. \tag{3.9}$$

Let $t = \log b = x^\varepsilon$. Since $b \ge 1$, $\log(1+b^2) \le 2t + \log 2$. Since $f \ge 1$ and $a \ge xb$, one has $4x^{99}/\mathcal{B} \le 4x^{98}e^{-t}t^{-3} \to 0$. The right side of (3.9), divided by $\mathcal{B}$, is therefore at most $24t + 10\log 2 + o(1) \le t^2$ for all sufficiently large x. This proves (3.8). □

The estimate remains valid when the individual volumes are extremely large or small. Their main size is represented by H, and is cancelled algebraically before any absolute upper bound is applied. The very coarse estimate (3.4) is used only for the missing proportion $4b^{-2}$ of directions. This explains why almost-full spanning is sufficient and why a much stronger full spanning theorem would be unnecessary.

Proposition 3.4 (`eventually_primeError_le_rpow`)**.** *Hypothesis 3.1, for every* $0 < \varepsilon < 1/100$ *and fixed* M, d*, implies* (1.1) *for every* $\eta > 0$*.*

Proof. Divide (3.8) by maS and use (2.8). Since $f/S = \varphi(M)T/S \le \varphi(M)\log b$, one obtains

$$R_{d,M}(x) \le \frac{\varphi(M)(\log b)^6}{m} + |R_{d,M}(y)| + \log b\,|R_{d,1}(y)|. \tag{3.10}$$

Fix $\eta > 0$ and choose $\varepsilon < \min(1/100, \eta/12)$. The first term is at most $\varphi(M)x^{-1/2+6\varepsilon}$. Lemma 2.2, with $\log y = x^{\varepsilon/6}$, makes the remaining terms $O_{M,d,\varepsilon}(x^{-2})$: choose its exponent A so that $A\varepsilon/6 > 2+\varepsilon$. Finally $6\varepsilon < \eta$, so the fixed coefficient and the error can be absorbed into $x^{\eta-6\varepsilon}$. This proves (1.1) for real x. □

This use of a distant cutoff is already compatible with the readable edition of the source. The compact edition instead proves a localized small-value result for transforms of prime errors and moves the second cutoff into a suitable window. For a human exposition permitting the classical fixed-modulus prime number theorem, (3.10) gives a shorter conclusion and requires no new localization theorem.

4. Short forms and their integral index

The short-form argument has two distinct parts. The arithmetic construction produces divisibility at the finite places, with a controlled loss of index, and an upper bound at the infinite places. A single rational scalar then removes the finite-place gains. The second part is elementary and is given in full here. Keeping it separate makes the exact arithmetic assertion that must be proved especially clear.

Write V for the space of columns, $V^{\mathrm{ram}} \subseteq V$ for its ramified subspace, W for the row space, and $d_V = \dim V$. Thus $V^{\mathrm{ram}} = \widehat{V}$ and $d_V = C$ in Section 3; throughout this section $x_1 = y$. The finite group acting on rows is denoted by Γ. All spaces and maps are over a common coefficient field equipped with the complex and p-adic embeddings in use. One may equivalently work with the complex spaces of the source and its chosen field embeddings into $\mathbf{C}_p$. For each such embedding v above p, normalize $|p|_v = p^{-1}$ and put $\mathcal{O}_v = \{z : |z|_v \le 1\}$. Let $M_v \subset V$ denote the module of columns integral at v, and put $M^{\mathrm{ram}} = V^{\mathrm{ram}} \cap \bigcap_v M_v$.

The admissible twists ϑ are the weight-zero tame characters of the construction whose first m positive powers remain nontrivial at every place in their tame support. Their support Q is disjoint from its conjugate, and the product $q_\vartheta = \prod_{w\in Q} \mathrm{N}\, w$ is prime to the auxiliary level gN. The trivial twist has empty support. For each twist let $\Theta_\vartheta : W \longrightarrow E_\vartheta$ be the terminal lift and $L_{\vartheta,v} \subset E_\vartheta$ its integral module. At the infinite places E_1 carries the squared class-sum norm $\|\cdot\|_\sigma^2$. The pointwise squared norm on columns is written $\|\cdot\|_{\infty,\sigma}^2$.

For modules over $\mathcal{O}_v$ we use a determinant formulation of index. We write $[M : S]_{\det} \mid p^r$ when every finite family $z_1, \dots, z_n \in M$ admits a matrix $U \in M_n(\mathcal{O}_v)$ such that $\det U$ divides p^r in $\mathcal{O}_v$ and $\sum_j U_{ij} z_j \in S$ for every i. In particular, $|\det U|_v \ge p^{-r}$. This definition is meaningful over the nondiscrete valuation ring of $\mathbf{C}_p$; no finite cardinality of a quotient is being asserted. The case $n = 0$ uses determinant 1.

For a finite family $y = (y_1, \dots, y_n)$ in W, say that its integral index is at most δ if there are nonnegative real numbers ϵ_p with $\sum_p \epsilon_p \le \delta$ such that, for every $v \mid p$ and every admissible ϑ with $(p, q_\vartheta) = 1$, some $U \in M_n(\mathcal{O}_v)$ satisfies $|\det U|_v \ge e^{-\epsilon_p}$ and $\Theta_\vartheta(\sum_j U_{ij} y_j) \in L_{\vartheta,v}$ for all i. The matrix is allowed to depend on v and ϑ. The numbers ϵ_p are common to all of them.

Set $\mathcal{B} = fa(\log b)^3$, $H = mk\mathfrak{u}$, where k is the positive source rank and $\mathfrak{u}$ is the direction cost, and let $\mathcal{R}$ denote the prime-sum remainder. The construction supplies maps $A_s : V^{\mathrm{ram}} \longrightarrow W$, $s \in \Gamma$, obtained by projecting the translates of a Poincaré series. More precisely, $A_s = |\Gamma| e_0 s \mathcal{P} \circ (c\,\mathrm{incl})$, where $\mathcal{P}$ is the Poincaré map, e_0 is the projector to W, and $c \ne 0$ is independent of the twist. The estimates required below are the following quantitative hypotheses.

There is a set $S_0 \subset M^{\mathrm{ram}}$ spanning V^{ram} such that $\|\phi\|_{\infty,\sigma}^2 \le e^{\mathcal{B}}$ for all $\phi \in S_0$ and all σ. There are finitely supported real numbers $g_p \ge 0$ and $e_p \ge 0$, and a real number h_0, satisfying

$$\sum_p e_p \log p \le \mathcal{B}, \qquad h_0 \le H - m\mathcal{R} + \mathcal{B}, \qquad x^{99} + 5\mathcal{B} \le (\log b)\mathcal{B}. \tag{4.1}$$

For every $\phi \in V^{\mathrm{ram}}$ whose pointwise squared norms are at most e^B, the projected chain has

$$\|\Theta_1 A_s \phi\|_\sigma^2 \le \exp\left(B + h_0 + 2\sum_p g_p \log p + \mathcal{B} + x^{99}\right) \quad (s \in \Gamma). \tag{4.2}$$

Finally, for every p, every $v \mid p$, and every integer $\beta < g_p - e_p + 1$, there exist a nonnegative integer $r \le e_p d_V$ and an $\mathcal{O}_v$-module $S_v \subset V$ such that

$$[M_v : S_v]_{\det} \mid p^r, \qquad p^{-\beta}\Theta_\vartheta A_s(S_v \cap V^{\mathrm{ram}}) \subseteq L_{\vartheta,v} \quad ((p, q_\vartheta) = 1, s \in \Gamma). \tag{4.3}$$

Here S_v is chosen before ϑ and s. This uniformity, together with the use of the ambient dimension d_V, is part of the hypothesis.

Proposition 4.1 (`Data.short_forms`)**.** *Under* (4.1)–(4.3)*, there is $q \in \mathbf{Q}_{>0}$ such that the maps $B_s = qA_s$ have the following properties. The set S_0 spans V^{ram} and, for every $\phi \in S_0$, $s \in \Gamma$, and complex embedding σ,*

$$\|\Theta_1 B_s \phi\|_\sigma^2 \le \exp(H - m\mathcal{R} + (\log b)\mathcal{B})\,. \tag{4.4}$$

For every finite $T \subseteq \Gamma$, every finite family contained in $\bigcup_{s\in T} B_s(M^{\mathrm{ram}})$ has integral index at most $|T| d_V \mathcal{B}$.

Proof. Put $\beta_p = \lceil g_p - e_p \rceil$ and $q = \prod_p p^{-\beta_p} = \prod_p p^{\lfloor e_p - g_p \rfloor}$. The product is finite, $q > 0$, and

$$|q|_v = p^{\beta_p} \quad (v \mid p), \qquad \log q \le \sum_p (e_p - g_p)\log p. \tag{4.5}$$

There is no additional rounding term in this choice. Since q is rational, every complex embedding fixes it. Applying (4.2) to $q\phi$, for $\phi \in S_0$, gives

$$\log \|\Theta_1 B_s \phi\|_\sigma^2 \le h_0 + 2\log q + 2\sum_p g_p \log p + 2\mathcal{B} + x^{99} \le H - m\mathcal{R} + 5\mathcal{B} + x^{99}.$$

If the vector vanishes, the asserted norm inequality holds directly; the display is otherwise its logarithmic form. Equation (4.4) follows from (4.1).

Fix T and a family $y_1, \dots, y_n$ in its indicated union, and choose $s(i) \in T$ and $\phi_i \in M^{\mathrm{ram}}$ with $y_i = B_{s(i)}\phi_i$. Fix p and $v \mid p$. Since $\beta_p < g_p - e_p + 1$, choose S_v and r as in (4.3). Partition the indices into the blocks $I_s = \{i : s(i) = s\}$. For every nonempty block, the determinant-index condition gives a matrix U_s carrying its source vectors into S_v and satisfying $\det U_s \mid p^r$. Each transformed source vector also lies in V^{ram}, because the latter is a vector subspace. Taking the block diagonal matrix U therefore gives $|\det U|_v \ge p^{-|T|r} \ge p^{-|T| d_V e_p}$.

The rational number qp^{β_p} is a unit of $\mathcal{O}_v$. Consequently (4.3) shows that every row of the transformed family has integral lift, simultaneously for all admissible twists prime to p. Taking $\epsilon_p = |T| d_V e_p \log p$ proves the index assertion, because (4.1) bounds their sum. □

The argument deliberately uses only a partition into images of the maps B_s. There is no need here for a general calculus of indices of sums of submodules. Moreover, the local construction above allows the matrix to be chosen independently of the twist once the source preimages are fixed, although the stated index condition does not require that strengthening.

Two elementary estimates explain the aggregate loss budget and the choice of the level. Write $\chi_K = \chi_d$ for the quadratic character of $K = \mathbb{Q}(\sqrt{-d})$; the finite gains g_p are those in the local hypotheses above.

Put $\ell_b = \log b$, $t = \log x_1$, and

$$\lambda_p = \frac{fa\ell_b^2}{p}\begin{cases} m, & p \equiv 1 \text{ or } -1 \pmod m, \\ 1, & \text{otherwise,}\end{cases} \qquad e_p = \begin{cases} 3\lambda_p, & p < x_1,\ \chi_K(p) \ne 1, \\ 4mfb, & p = N, \\ 0, & \text{otherwise.}\end{cases}$$

The level prime satisfies $N > x_1$, so the cases are disjoint.

Lemma 4.2. *Suppose $m \in \mathbb{N}$, $m \ge 2$, $m^2 \le 4x$, $a \ge xb$, $\log N \le 30m\ell_b$, $t \ge 4$, and $\ell_b \ge t^6$. Then $\sum_p e_p \log p \le fa\ell_b^3$.*

Proof. The elementary bound $\sum_{j\le X} j^{-1} \le 1+\log X$ gives $\sum_{p<x_1}(\log p)/p \le t(1+t)$. For primes congruent to 1 or -1 modulo m, write $p=mc+1$ or $p=mc-1$, respectively. Then $m/p \le 1/c$ or $m/p \le 2/c$, and $1 \le c \le x_1$. Their extra contribution is therefore at most $3t(1+t)$. Consequently the nonsplit primes contribute at most $12fa\ell_b^2 t(1+t) \le \frac{1}{2}fa\ell_b^3$; the last inequality follows from $24t(1+t) \le t^6 \le \ell_b$ for $t \ge 4$. The level prime contributes at most $4mfb\log N \le 120m^2fb\ell_b \le 480fa\ell_b \le \frac{1}{2}fa\ell_b^3$, since $\ell_b \ge 4^6$ implies $\ell_b^2 \ge 960$. □

For the infinite-place cancellation, let $a \in \mathbb{N}_{\ge 1}$ and let $\ell = \ell_d$ be the constant defined in Section 2. Let E be the auxiliary CM field, let $\mathrm{rd}_E = |\operatorname{Disc}(E)|^{1/[E:\mathbb{Q}]}$ be its root discriminant, and let $g>0$ be the auxiliary level factor. Let $\mathcal{T}$ and J_τ be as in Lemma 2.3, with $\mathrm{Ask}(a,\tau)=J_\tau$. The f nominal weights κ_σ equal a/x at S thick places and a/x_1 elsewhere. Put $\bar w = f^{-1}\sum_\sigma \log\kappa_\sigma$ and $G=\log(g\ \mathrm{rd}_E/(2\pi))$, and use the period normalization $\log(\Omega/g)=\ell-G$. Define $s_\infty(j)=\log(j!)-j\log(\Omega/g)$ and $A_0=a(\log a-1-\ell-\bar w)$. If $\mathcal{R}=\sum_{\tau\in\mathcal{T}}(\mathrm{Ask}(a,\tau)-A_0)$, then the expression for h_0 is

$$h_0 = H - m\mathcal{R} + |\mathcal{T}|(m(s_\infty(a)-A_0)-a\log N).$$

Integrating the chord inequality for the concave function $\log t$ on each unit interval gives $\log(a!) \le a\log a - a + 1 + \frac{1}{2}\log a$. Combining this with the level choice $\log N \ge m(\bar w+G)-G-\frac{15}{2}\ell_b$ yields

$$h_0 \le H - m\mathcal{R} + |\mathcal{T}|\left(m(1+\tfrac{1}{2}\log a) + a(G+\tfrac{15}{2}\ell_b)\right). \tag{4.6}$$

Thus verifying $h_0 \le H - m\mathcal{R} + \mathcal{B}$ reduces to bounding the displayed residual by $\mathcal{B}$. This residual bound is included in the arithmetic normalization assumed in (4.1). The key cancellation is the exact removal of $ma(\bar w+G)$ by $a\log N$. This is why the level is chosen from that window.

The unresolved arithmetic content of (4.2) and (4.3) should not be conflated with the elementary proof above. It can be stated precisely as follows. A controlled seesaw identity must provide one constant $c\neq 0$, independent of ϑ, and for each twist a finite combination Ψ_ϑ of theta data such that

$$\iota_\vartheta \mathcal{P}(c\phi) = C_\vartheta^{\Psi_\vartheta}\phi. \tag{4.7}$$

Its coefficients must be integral away from q_ϑ, its data must have the stipulated joint invariance, and for the trivial twist its total coefficient size must be at most $e^{x^{60}}$ at every complex embedding. One then needs the pointwise bound

$$\|C_1^{\Psi_1}\phi\|_{\infty,\sigma}^2 \le \exp\left(B + h_0 + 2\sum_p g_p\log p + \mathcal{B}\right) \quad \text{if } \|\phi\|_{\infty,\sigma}^2 \le e^B. \tag{4.8}$$

The projection $|\Gamma|e_0 s$ preserves the integral modules and costs at most $e^{x^{99}}$ in squared class-sum norm, which converts (4.8) into (4.2).

The local assertion concerns the same data and the same constant. Its critical case is a nonsplit prime at which two top weights are joined by Frobenius and conjugation. The construction selects split places over p, one from each required conjugate pair, and forms a product Y_p of scaled Hecke operators. Put $j_p=\lceil 2mfa/p\rceil$. The two estimates needed in this case are

$$\begin{aligned} &Y_pM_v \subseteq M_v, \qquad [M_v : Y_p^m M_v + p^{j_p}M_v]_{\det} \mid p^{r_p}, \quad r_p \le 3\lambda_p d_V,\\ &\|C_\vartheta^\Psi(Y_p^m\phi)\|_v \le p^{2\lambda_p-g_p}, \qquad \|C_\vartheta^\Psi(p^{j_p}\phi)\|_v \le p^{\lambda_p-g_p} \qquad (\phi\in M_v). \end{aligned} \tag{4.9}$$

The norm here is the supremum of the p-adic coefficient gauges. These estimates must hold for every good datum Ψ and every twist prime to p. Their consequence is elementary: take $S_v = Y_p^m M_v + p^{j_p}M_v$. The ultrametric inequality gives gauge at most $p^{2\lambda_p-g_p}$ on S_v. Since $\lambda_p \ge 1$ and $\beta < g_p - 3\lambda_p + 1$, one has $2\lambda_p - g_p \le -\beta$. Multiplication by $p^{-\beta}$ therefore makes the chain integral, with the required determinant-index bound.

For the other small nonsplit primes, the source uses a direct collision bound: the chain gauge is at most $p^{mc_p+\lambda_p-g_p}$, and $mc_p \le \lambda_p$. The same reasoning then works with $S_v=M_v$ and index exponent 0. At $p=N$, a finite Fourier operator F_N gives a submodule of determinant index dividing N^{r_N}, $r_N \le 4mfb\,d_V$, on which the chain gauge is at most N^{3mfb}. Since $mfb \ge 1$ and

$\beta < -4mfb + 1$, this again yields the asserted divisibility. At all remaining primes the chain preserves integrality directly, so that $S_v = M_v$ and $\beta \le 0$. Equation (4.7) and integrality of the projected translations transfer these conclusions to (4.3).

The construction of Y_p and its small determinant index in (4.9), the uniform denominator control in (4.7), and the precise jet estimates behind (4.8) are specialized assertions of the proposed argument. General theta correspondence, ordinary Hecke integrality, or the finiteness of algebraic modular-form spaces does not by itself supply these inequalities. Until their full proofs have been incorporated, the proposition in this section is a proved local-to-global implication with the indicated arithmetic hypotheses.

5. Second moments and spanning

The spanning argument has two parts with very different mathematical content. The arithmetic construction must provide an almost isometric Poincaré map and a second moment with leading constant one. Once these estimates are available, a finite-dimensional argument chooses a small collection of translates spanning almost all the target. The latter argument uses the relation between second moments and nearly tight frames, followed by greedy covering; these ideas have classical precedents in [1, 5]. We give the argument in full, both to keep track of the constants and to isolate the arithmetic estimate that cannot be replaced by a general reference to those precedents.

Let V and H be finite-dimensional complex Hilbert spaces, let $c = \dim V > 0$, and let a finite group Γ act unitarily on H through ρ. Let R be an orthogonal projection commuting with this action, put $W = RH$, and write $n = \dim W$. For a subspace U of a Hilbert space, Π_U denotes orthogonal projection onto U. We use the Hilbert–Schmidt norm $\|A\|_{\mathrm{HS}}^2 = \operatorname{tr}(A^*A)$.

Lemma 5.1 (Second moments control preimages)**.** *Suppose that $n > 0$ and that $B : V \longrightarrow H$ satisfies, for $\delta, \varepsilon \ge 0$,*

$$\|RB\Pi_U\|_{\mathrm{HS}}^2 \ge \dim U - \delta c \qquad (U \le V), \tag{5.1}$$

$$n \sum_{s\in\Gamma} \|B^*\rho(s)B\|_{\mathrm{HS}}^2 \le |\Gamma|(1+\varepsilon)c^2. \tag{5.2}$$

For every subspace $F \le W$, with $\alpha = \dim F/n$, there exists $s \in \Gamma$ such that

$$\dim\bigl(R\rho(s)B\bigr)^{-1}F \le \left(\frac{1+\alpha}{2} + \delta + \frac{\varepsilon}{2}\right) c. \tag{5.3}$$

Proof. Put $X_s = \rho(s)BB^*\rho(s)^*$ and $X = \sum_s X_s$. Cyclicity of the trace and unitarity give

$$\langle X_s, X_t\rangle_{\mathrm{HS}} = \|B^*\rho(s^{-1}t)B\|_{\mathrm{HS}}^2, \qquad \|X\|_{\mathrm{HS}}^2 = |\Gamma| \sum_s \|B^*\rho(s)B\|_{\mathrm{HS}}^2.$$

Since $\|\Pi_F\|_{\mathrm{HS}}^2 = \dim F$, Cauchy–Schwarz applied to $\operatorname{tr}(\Pi_F X)$ implies

$$\frac{1}{|\Gamma|}\sum_s \|\Pi_F\rho(s)B\|_{\mathrm{HS}}^2 \le c\sqrt{\alpha(1+\varepsilon)} \le \frac{1+\alpha+\varepsilon}{2}\, c.$$

Choose s for which the summand does not exceed this upper bound for its average, and set $U = (R\rho(s)B)^{-1}F$. On U, the maps $R\rho(s)B$ and $\Pi_F\rho(s)B$ agree: the component $(I-R)\rho(s)Bx$ is orthogonal to F, while $R\rho(s)Bx$ already belongs to F for $x \in U$. Because R commutes with $\rho(s)$ and $\rho(s)$ is unitary,

$$\dim U - \delta c \le \|RB\Pi_U\|_{\mathrm{HS}}^2 = \|\Pi_F\rho(s)B\Pi_U\|_{\mathrm{HS}}^2 \le \|\Pi_F\rho(s)B\|_{\mathrm{HS}}^2.$$

The last inequality follows by extending an orthonormal basis of U to one of V. Combining the two estimates proves (5.3). □

The first hypothesis is weaker than an operator norm estimate. For example, $\|B^*RB - I\|_{\mathrm{HS}} \le \delta\sqrt{c}$ implies (5.1), because $|\operatorname{tr}((B^*RB - I)\Pi_U)| \le \delta\sqrt{c\dim U} \le \delta c$. This observation explains why a mean-square estimate suffices: one need not control each singular value separately.

For a family $A_s : V \longrightarrow W$ and $\eta > 0$, say that the family *spreads with error* η if every $F \leq W$ admits an index s with $\dim A_s^{-1}F \leq ((1 + \dim F/n)/2 + \eta)c$. This is precisely the property needed by the greedy choice below; no choice of a basis or of a filtration is involved.

Lemma 5.2 (A small set of translates)**.** *Let $c = \dim V > 0$, $n = \dim W > 0$, and let $(A_s)_{s\in\Theta}$ spread with error $\eta > 0$. There is a finite set $T \subseteq \Theta$ such that*

$$|T|c \leq 5n\log(1 + \eta^{-1}), \qquad \dim \sum_{s\in T} A_s(V) \geq (1 - 4\eta)n. \tag{5.4}$$

Proof. If $\eta \geq 1/4$, take $T = \varnothing$. Otherwise begin with $F = 0$ and, while its codimension $d = n - \dim F$ exceeds $4\eta n$, choose an index supplied by spreading and replace F by $F + A_s(V)$. The gain Δ in dimension is the rank of $V \xrightarrow{A_s} W \longrightarrow W/F$, hence

$$\Delta = c - \dim A_s^{-1}F \geq c\left(\frac{d}{2n} - \eta\right) > 0. \tag{5.5}$$

In particular this index has not already been selected, and the process stops after finitely many steps. To bound their number, use the potential $\Phi(d) = 5n\log(1 + d/(\eta n))$. Since $0 < \Delta \leq d$ and $d > 4\eta n$, the inequality $-\log(1 - u) \geq u$ for $0 \leq u < 1$ gives

$$\Phi(d) - \Phi(d - \Delta) \geq \frac{5n\Delta}{d + \eta n} \geq 5c\frac{d/2 - \eta n}{d + \eta n} \geq c.$$

Thus every selected index costs at most the decrease in the potential divided by c. Summing gives $|T|c \leq \Phi(n) - \Phi(d_{\text{final}}) \leq 5n\log(1 + \eta^{-1})$. The stopping condition gives the dimension bound. □

The logarithmic potential is useful here: a bound by geometric decay alone usually introduces a rounding term proportional to c. The potential proves the exact bound in (5.4), including its constant, without such a term.

We also record explicitly how ramification changes the argument. For an endomorphism E of V and a subspace $U \leq V$, the sequence $0 \longrightarrow \ker E \longrightarrow E^{-1}U \xrightarrow{E} U \cap E(V) \longrightarrow 0$ is exact. Consequently $\dim E^{-1}U \leq \dim U + \dim\ker E$. In particular, if $E = \prod_{j=1}^{f}(I - Z_j)$ and $\dim\ker(I - Z_j) \leq c/(N - 1)$ for every j, then

$$\dim E^{-1}U \leq \dim U + \frac{f}{N - 1}\,c. \tag{5.6}$$

Indeed, applying the preceding inequality successively proves this bound; commutativity of the factors is unnecessary for the dimension estimate. If (A_s) spreads with error η_0, then (A_sE) therefore spreads with error at most $\eta_0 + f/(N - 1)$.

Proposition 5.3 (`Data.enough_forms`)**.** *Suppose $c > 0$, $b \geq 4$, $N > 1$, and $\mathcal{M} > 0$. In the notation above, assume the following arithmetic estimates:*

$$n \leq |\Gamma|\mathcal{M}, \tag{5.7}$$

$$\|RB\Pi_U\|_{\text{HS}}^2 \geq \dim U - 5b^{-4}c \qquad (U \leq V), \tag{5.8}$$

$$\sum_{s\in\Gamma}\|B^*\rho(s)B\|_{\text{HS}}^2 \leq (1 + 9b^{-4})\frac{c^2}{\mathcal{M}}. \tag{5.9}$$

Let $E : V \longrightarrow V$ satisfy (5.6)*, and assume*

$$\frac{19}{2}b^{-4} + \frac{f}{N - 1} \leq b^{-2}. \tag{5.10}$$

Put $\widehat{V} = E(V)$ and, for any fixed $\lambda \neq 0$, define $Q_s = \lambda R\rho(s)B|_{\widehat{V}} : \widehat{V} \longrightarrow W$. Then $n > 0$, and there is a set $T \subseteq \Gamma$ such that

$$|T|c \leq 5n\log(1 + b^2), \qquad \dim \sum_{s\in T} Q_s(\widehat{V}) \geq (1 - 4b^{-2})n. \tag{5.11}$$

Proof. Taking $U = V$ in (5.8) shows that $RB \neq 0$, since $5b^{-4} < 1$; hence $n > 0$. Multiplication of (5.9) by (5.7) gives (5.2) with $\varepsilon = 9b^{-4}$. Lemma 5.1, with $\delta = 5b^{-4}$, shows that the maps $R\rho(s)B$ spread with error $19b^{-4}/2$. Equations (5.6) and (5.10) give error b^{-2} after composing with E. Apply Lemma 5.2 to these composites. Finally, $R\rho(s)BE(V) = R\rho(s)B(\widehat{V})$, and multiplication by $\lambda \neq 0$ does not change a linear image. This gives (5.11). □

Here V is the full column space, W is the row space, Γ is the finite level group, R is the projector to the good types, and B is the normalized Poincaré map. In the source construction $\mathcal{M}$ is the product of the relevant formal degrees with the mass of the definite genus. Notice that the cost $|T|c$ uses the dimension of the *full* column space, even though Q_s is defined on the ramified space $\widehat{V}$. This distinction is needed later when the short columns are used in a determinant.

The point requiring a construction-specific proof is (5.9), together with the projection estimate (5.8). A general theorem asserting the existence or spanning property of Poincaré series does not give either bound. The leading constant 1 in (5.9), the uniform $9b^{-4}$ error, and the loss $5b^{-4}$ after deleting the bad types are the quantitative content. In the compact source their proof proceeds through local Bessel kernels, bounds for the energy on vector-fixing sets, and a sieve for nonnegative functions of positive type. Proposition 5.3 proves the deduction from these explicit arithmetic hypotheses, with the constants required by the covolume comparison.

6. Determinants and the lower bound

The lower bound is an application of the product formula to two reciprocal families of Gram determinants. Its elementary part is short. The substantial arithmetic assertion is the existence of twists for which both determinant bounds hold with the stated total error. We give a complete proof of the determinant argument and state that arithmetic assertion explicitly. In particular, the product formula does not by itself supply a lower bound for the volume of an arbitrary space of automorphic forms.

Write D for the dimension of the row space, and put

$$\mathcal{B} = fa(\log b)^3, \qquad L = (\log b)\mathcal{B}, \qquad B_0 = \frac{\mathcal{B}}{mk}, \qquad H = mk\mathfrak{u}. \tag{6.1}$$

Here m, k are positive integers, $f, a > 0$, $b > 1$, and $\mathfrak{u}$ is the direction cost; the main term is H. Only the identities in (6.1) and the indicated numerical inequalities will be used in this section.

For each rational prime p, let $\mathcal{I}_p$ be the set of abstract field isomorphisms $\mathbb{C} \simeq \overline{\mathbb{Q}_p}$, and set $\mathcal{I} = \coprod_p \mathcal{I}_p$. These are field isomorphisms, with no continuity requirement. We normalize the absolute value by $|p|_p = p^{-1}$. The field $\overline{\mathbb{Q}_p}$ is an algebraic closure of $\mathbb{Q}_p$, equipped with its extended absolute value and its inclusion in $\mathbb{C}_p$. Composing with that inclusion gives the embeddings used in the integral index condition. Such isomorphisms exist: both fields are algebraically closed of characteristic zero and of transcendence degree $|\mathbb{R}|$ over $\mathbb{Q}$. An embedding of a number field in either field extends to one of the isomorphisms in question.

Definition 6.1. For a family $y = (y_\iota)_{\iota \in \mathcal{I}}$ of complex numbers, the notation $h_{\mathrm{f}}(y) \leq C$ means that there are $A \in \mathbb{C}^\times$, $\alpha \in \mathbb{R}$, and real numbers e_p such that $|\sigma A| \leq e^\alpha$ for every $\sigma \in \mathrm{Aut}(\mathbb{C})$, $|\iota(Ay_\iota)|_p \leq e^{e_p}$ for every $\iota \in \mathcal{I}_p$, and $\alpha + \sum_{p<M} e_p \leq C$ for every sufficiently large integer M.

This notation denotes a bound with a witness, rather than an assertion that an infimum is attained. The family may vary with the prime and its embedding; h_{f} is therefore not the usual nonnegative Weil height of a single algebraic number. Negative bounds are possible. Multiplication adds bounds: multiply the witnesses and add their archimedean and finite place bounds. Consequently finite products and nonnegative integral powers satisfy the corresponding subadditivity inequalities. If $|\iota z_\iota|_p \leq |\iota y_\iota|_p$ for every ι, the same witness for y works for z.

Lemma 6.2. *If $h_{\mathrm{f}}(1) \leq C$, then $C \geq 0$.*

Proof. Let A, α, e_p be a witness. The bound on all complex conjugates implies that A is algebraic. Indeed, if A were transcendental, for an arbitrarily large transcendental $t \in \mathbb{C}$ the map $\mathbb{Q}(A) \to \mathbb{Q}(t)$ taking A to t would extend, by choosing transcendence bases and

then algebraic closures, to an automorphism of $\mathbb{C}$. This contradicts the bound on $|\sigma A|$. Let $K = \mathbb{Q}(A)$, $N = [K : \mathbb{Q}]$, and $q = N_{K/\mathbb{Q}}(A) \neq 0$. Every embedding of K into $\mathbb{C}$ or $\overline{\mathbb{Q}_p}$ extends to one occurring in the witness. Multiplying the N conjugate bounds gives $N^{-1} \log |q| \leq \alpha$ and $N^{-1} \log |q|_p \leq e_p$. Once M includes all prime divisors of the numerator and denominator of q, the rational product formula gives $0 \leq \alpha + \sum_{p<M} e_p \leq C$. □

An elementary point about complex conjugation matters here. An arbitrary automorphism σ of $\mathbb{C}$ need not commute with complex conjugation c. It is therefore incorrect to treat $\sigma(\operatorname{Gram}(x))$ as the positive Gram matrix of σx. For vectors represented in a space with rational positive definite Gram matrix, put $\sigma' = c\sigma c$. Then $\sigma \operatorname{Gram}(x)$ is the mixed Gram matrix of $\sigma' x$ and σx. The Cauchy–Schwarz inequality in the top exterior power gives

$$|\sigma \det \operatorname{Gram}(x)| \leq \det \operatorname{Gram}(\sigma' x)^{1/2} \det \operatorname{Gram}(\sigma x)^{1/2}. \tag{6.2}$$

The same argument applies to a finite direct sum with positive rational weights, as in the class-sum inner products of definite automorphic forms. In particular, a bound e^{α} for the squared covolume at every complex conjugate implies $|\sigma \det \operatorname{Gram}(x)| \leq e^{\alpha}$ for every σ.

Let W be a D-dimensional complex Hilbert space, let o be a distinguished twist, and let $u_\vartheta : W \to H_\vartheta$ be injective linear maps to finite-dimensional complex Hilbert spaces. Put $Z_\vartheta = \det(u_\vartheta^* u_\vartheta) > 0$. For a basis x of W, its image has Gram determinant $G_\vartheta = Z_\vartheta \det \operatorname{Gram}_W(x)$, so the quotient $G_\vartheta / G_o = Z_\vartheta / Z_o$ is independent of x. Our Hermitian products are conjugate-linear in the first variable.

Proposition 6.3. *Fix a twist θ_ι for each $\iota \in \mathcal{I}$, a basis $x = (x_1, \dots, x_D)$, nonzero numbers γ_ι, and an integer $q \geq 0$. Suppose that $|\sigma G_o| \leq e^{\alpha}$ for every σ, $h_{\mathrm{f}}((\gamma_\iota \overline{\gamma_\iota})^{-1}) \leq g$, and there are real numbers e_p with $\sum_{p \in T} e_p \leq \delta$ for every finite set of primes T. For each $\iota \in \mathcal{I}_p$ and $j \in \mathcal{I}_p$, suppose there is a set $\Lambda_{\iota,j} \subset H_{\theta_\iota}$ satisfying*

$$|j\langle v, w\rangle|_p \leq 1 \quad (v \in \Lambda_{\iota, j\circ c},\ w \in \Lambda_{\iota,j}). \tag{6.3}$$

Suppose also that there is a matrix $M_j \in M_D(\mathbb{C})$ such that $\sum_a (M_j)_{ra} u_{\theta_\iota} x_a \in \Lambda_{\iota,j}$ for every r and $|j \det M_j|_p \geq e^{-e_p} |j\gamma_\iota|_p^q$. Then

$$h_{\mathrm{f}}(Z_{\theta_\iota} / Z_o) \leq \alpha + 2\delta + qg. \tag{6.4}$$

Proof. Fix $\iota \in \mathcal{I}_p$ and use the two matrices M_ι and $M_{\iota\circ c}$ for this same twist θ_ι. The mixed Gram matrix of their recombined vectors is $\overline{M_{\iota\circ c}}\, G_{\theta_\iota}^{\mathrm{mat}} M_\iota^t$, where $G_{\theta_\iota}^{\mathrm{mat}}$ denotes the Gram matrix, rather than its determinant. Every entry is ι-integral by (6.3); the ultrametric inequality therefore bounds its determinant by 1. The two lower bounds for the determinants of the recombination matrices give

$$\left|\iota\left((\gamma_\iota \overline{\gamma_\iota})^q G_{\theta_\iota}\right)\right|_p \leq e^{2e_p}.$$

Take the common witness $A = G_o \neq 0$. The equality $G_o Z_{\theta_\iota} / Z_o = G_{\theta_\iota}$ shows that $(\gamma_\iota \overline{\gamma_\iota})^q Z_{\theta_\iota} / Z_o$ has height at most $\alpha + 2\delta$. Multiply by the qth power of the inverse scalar family. This proves (6.4). □

The factor 2δ is forced by the two conjugate recombinations. Their matrices may depend on both the selected twist and the embedding. One must not replace θ_ι by $\theta_{\iota\circ c}$ when taking the second matrix. The integral index condition has quantifier order "for each place and each allowed twist, there exists a matrix." It does not assert that one matrix works for every twist. Proposition 6.3 requires only the former order.

Here is the completion argument used in the application. Let $y_1, \dots, y_n \in W$ be independent, and let S span W. Suppose that the squared covolume of $(u_o y_a)$ is at most e^V at every conjugate, and that each $u_o z$, $z \in S$, has conjugate squared norm at most e^A. Choose $q = D - n$ vectors from S completing y to a basis. Fischer's determinant inequality, or orthogonal projection followed by Hadamard's inequality, bounds the squared covolume of the full basis by e^{V+qA}. Equation (6.2) supplies the bound on all conjugates of its Gram determinant required above. Suppose the y have index at most δ: for each allowed twist and each $j \in \mathcal{I}_p$, an integral $n \times n$ matrix with determinant of absolute value at least e^{-e_p} recombines their images into the j-integral forms.

Suppose also that $\gamma_\iota u_{\theta_\iota} z$ is integral for every $z \in S$ and every $j \in \mathcal{I}_p$. Extending the index matrix by the block $\gamma_\iota I_q$ verifies Proposition 6.3, and hence

$$h_{\mathrm{f}}(Z_{\theta_\iota}/Z_o) \le V + 2\delta + (D-n)(A+g). \tag{6.5}$$

In particular, if $n \ge (1-4/b^2)D$ and $A+g \le b^3/4$, the last term is at most Db.

We now isolate the precise arithmetic input. All lattices below are equipped with the conjugate integral pairing (6.3), and untwisted forms have the rational coordinate realizations used in (6.2). There is a family of twists θ_ι, with conductor prime to the residue characteristic of ι, for which the following hold. These are members of the same admissible twist family as in Section 4. First, the completion data just described exist, with $A+g \le b^3/4$. Second, there are nonzero $c(\vartheta)$ and positive single-lift determinants $\lambda_j(\vartheta)$, $1 \le j \le m$, such that

$$Z_\vartheta = |c(\vartheta)|^D \prod_{j=1}^{m} \lambda_j(\vartheta), \qquad h_{\mathrm{f}}\big(|c(\theta_\iota)/c(o)|^{-2}\big) \le 2L. \tag{6.6}$$

Third, for each j the same selected family satisfies

$$h_{\mathrm{f}}\big(\lambda_j(o)/\lambda_j(\theta_\iota)\big) \le Dk\big((\log b)B_0 - \mathfrak{u}\big). \tag{6.7}$$

These are hypotheses about this automorphic construction, not general properties of theta lifts. In the source $o = 1$, $u_\vartheta = \Theta_\vartheta$, and $\lambda_j(\vartheta) = \det(u_{j,\vartheta^j}^* u_{j,\vartheta^j})$: the jth power of the twist is already included in this notation.

Theorem 6.4 (`Data.integral_forms_long`; conditional determinant form)**.** *Assume the preceding arithmetic input and $b \le L$. If $n \ge (1-4/b^2)D$ independent rows have index at most δ and untwisted squared covolume at most e^V at every complex conjugate, then*

$$D(H-3L) \le V + 2\delta. \tag{6.8}$$

Proof. Equation (6.5) gives $h_{\mathrm{f}}(Z_{\theta_\iota}/Z_o) \le V + 2\delta + Db$. Adding (6.7) over the m lifts gives the bound $D(L-H)$ for $\prod_j \lambda_j(o)/\lambda_j(\theta_\iota)$. Squaring (6.6) and using its scalar bound therefore gives

$$h_{\mathrm{f}}(Z_o^2/Z_{\theta_\iota}^2) \le 2D(L-H) + 2DL = 2D(2L-H). \tag{6.9}$$

The pointwise product of this family with $(Z_{\theta_\iota}/Z_o)^2$ is 1. Lemma 6.2 thus yields $0 \le 2(V + 2\delta + Db) + 2D(2L-H)$, or $D(H-2L-b) \le V + 2\delta$. Since $b \le L$, this implies (6.8). □

The normalization is essential: Z_ϑ is already the *squared* volume of Θ_ϑ. The reciprocal family in (6.9) is a ratio of fourth powers of ordinary volumes. Replacing Z by an ordinary covolume without adjusting all factors would alter both the index cost and the main term.

The single-lift estimate can itself be separated into one transparent identity and arithmetic bounds. This gives a useful specification of what an independent proof of (6.7) must establish. Let U, T, V_ϑ be complex Hilbert spaces, $\dim U = D$, and suppose that $u_\vartheta : U \to V_\vartheta$ is injective, $g_\vartheta : T \to V_\vartheta$ is linear, and $A : T \to U$ is a fixed surjection. Assume $u_\vartheta^* g_\vartheta = \gamma_\vartheta A$, where $\gamma_\vartheta \ne 0$. For a fixed $\kappa_0 > 0$, set $L_\vartheta = \operatorname{im} u_\vartheta$, $C_\vartheta = \kappa_0 \operatorname{pr}_{L_\vartheta} g_\vartheta g_\vartheta^*|_{L_\vartheta}$, and $\lambda_\vartheta = \det(u_\vartheta^* u_\vartheta)$.

Lemma 6.5. *The operator C_ϑ is positive definite and*

$$\lambda_\vartheta \det C_\vartheta = \kappa_0^D |\gamma_\vartheta|^{2D} \det(AA^*). \tag{6.10}$$

If positive scalars Γ_ϑ satisfy $|\gamma_\vartheta/\gamma_o|^2 = \Gamma_\vartheta/\Gamma_o$, then

$$\frac{\lambda_o}{\lambda_\vartheta} = \frac{\det C_\vartheta}{\det C_o} \left(\frac{\Gamma_o}{\Gamma_\vartheta}\right)^D. \tag{6.11}$$

Proof. The map $g_\vartheta^* u_\vartheta = \overline{\gamma_\vartheta} A^*$ is injective, since A is surjective. Hence g_ϑ^* is injective on L_ϑ, proving positivity of C_ϑ. On U we have $u_\vartheta^* C_\vartheta u_\vartheta = \kappa_0 |\gamma_\vartheta|^2 AA^*$. Choose orthonormal bases of U and L_ϑ and take determinants; the determinant of $u_\vartheta^* C_\vartheta u_\vartheta$ is $\det(u_\vartheta^* u_\vartheta) \det C_\vartheta$. This proves (6.10). Divide the identity at o by the identity at ϑ and use the relation between γ and Γ to obtain (6.11). □

For the jth lift the repository constructs comparison forms g_ϑ of this kind. The scalar κ_0 is its small hook number, and Γ_ϑ is an explicitly normalized product of Hecke values. Let $r > 0$ be the auxiliary rank appearing in the integral comparison. The required arithmetic bounds are

$$|\sigma \det C_o| \le \exp(2DrB_0), \qquad |\iota \det C_{\theta_\iota}|_p \le \exp(Dr\beta_{j,p}), \tag{6.12}$$

$$\sum_{p\in S} \beta_{j,p} \le B_0, \qquad h_{\mathrm{f}}(\Gamma_o/\Gamma_{\theta_\iota}) \le 10mx^{999} + k(6B_0 - \mathfrak{u}). \tag{6.13}$$

where the third inequality is required for every finite set of primes S. The first two bounds, with the witness $\det C_o$, give $h_{\mathrm{f}}(\det C_{\theta_\iota} / \det C_o) \le 3DrB_0$. Equation (6.11) and (6.13) give

$$h_{\mathrm{f}}(\lambda_o/\lambda_{\theta_\iota}) \le D\{3rB_0 + 10mx^{999} + k(6B_0 - \mathfrak{u})\}.$$

Thus (6.7) follows from the explicit scale inequality $3rB_0 + 10mx^{999} + 6kB_0 \le k(\log b)B_0$. No passage from an individual eigenform estimate to a determinant estimate is implicit in this calculation.

What remains arithmetic is now sharply specified. The comparison forms must exist with the same surjection A for every twist; they must be integral at all embeddings prime to the twist conductor; and their images must satisfy the Hecke conditions needed to apply the denominator estimate to C_ϑ. A family of twists must meet these conditions simultaneously for every $1 \le j \le m$ and for the auxiliary rank-one pairs, with a summable budget $\sum_p \beta_{j,p} \le B_0$. The Hecke-value estimate (6.13), the chain factorization (6.6), and the auxiliary completion forms also require their specified normalizations. These are the construction-specific inputs recorded in Appendix A.

The product formula, Gram determinant inequalities, and the compressed operator identity are standard algebraic tools. The distinctive mechanism of this construction is that the twist can be chosen separately at each finite place, while the height witness remains a single untwisted determinant. Reciprocal determinant bounds then cancel the main direction cost. The repository treats the simultaneous twist, integral comparison, and budget estimates as new arguments; this is a statement about its proof organization, not an assertion that their originality has been established by a complete literature review. Citing a theta inner-product formula, an individual congruence bound, or an Iwasawa vanishing theorem does not establish the uniform package (6.6)–(6.13). They remain explicit hypotheses of Theorem 6.4.

7. A zero-free quadratic twist

The analytic argument needs one quadratic twist that is zero-free in a fixed closed rectangle, simultaneously for finitely many fixed Dirichlet characters. This is weaker than a uniform zero-density theorem in both conductor and height. Jutila's mean-value theorem allows arbitrary complex coefficients [2, Theorem 1, p. 192], whereas his stated zero-density theorem concerns real characters [2, Theorem 2, p. 193]. To make the passage to a fixed complex character explicit, we prove the weaker statement needed here. The proof uses only the Pólya–Vinogradov bound, the theta functional equation, and Hadamard's three-lines theorem; the standard character and theta facts may be found in [3].

For $j \in \{0, 1\}$ set $g_j(t) = t^j e^{-\pi t^2}$ and, for a sequence c with $|c(n)| \le 1$, define

$$\theta_{j,c}(t) = \sum_{n\ge 1} c(n) g_j(nt), \qquad J_{j,c}(T) = \int_{1/T}^{\infty} |\theta_{j,c}(t)| \, \frac{dt}{\sqrt{t}}.$$

These series and integrals converge absolutely for $t, T > 0$; for example, $|\theta_{j,c}(t)| \ll e^{-t}/t$. We write $J(a \mid n)$ for the Jacobi character in the numerator, with the principal local character at 2. Thus $J(a \mid n)$ is multiplicative in n, agrees with the usual Jacobi symbol for odd n, and its factor at an even denominator is 1 for odd a and 0 for even a. The character $v \mapsto J(-v \mid m)$ is nonprincipal unless m or $2m$ is a square. Consequently Pólya–Vinogradov gives

$$\left| \sum_{1\le v\le X} J(-v \mid m) \right| \ll \sqrt{m}\log(2m) \quad \text{if neither } m \text{ nor } 2m \text{ is a square.} \tag{7.1}$$

The implied constant is absolute; the trivial upper bound X applies in the remaining cases.

Lemma 7.1 (A smoothed mean value)**.** *For every $\eta > 0$, uniformly for $j \in \{0,1\}$, $T \geq 1$, $X \leq T^2$, $t \geq T^{-1}$, and $|c(n)| \leq 1$, one has*

$$\sum_{1 \leq v \leq X} \left|\theta_{j,c(\cdot)J(-v|\cdot)}(t)\right|^2 \ll_\eta T^2 t^{-1-2\eta}, \tag{7.2}$$

$$\sum_{1 \leq v \leq X} J_{j,c(\cdot)J(-v|\cdot)}(T) \ll_\eta T^{2+\eta}. \tag{7.3}$$

Proof. Expand the square in (7.2), using absolute convergence and $J(-v \mid m)J(-v \mid n) = J(-v \mid mn)$. For every $a > 0$ one has $g_j(u) \ll_a u^{-a}$ for $u > 0$. For the terms for which kmn is a square for $k = 1$ or 2, use $a = 1/2 + \eta$ and the trivial character-sum bound. Their contribution is at most a constant depending on η times

$$Xt^{-1-2\eta} \sum_{\substack{m,n \geq 1 \\ mn \text{ or } 2mn \text{ a square}}} (mn)^{-1/2-\eta} \ll_\eta T^2 t^{-1-2\eta}.$$

To justify convergence of the displayed sum, write $km = ar^2$ and $n = au^2$ with a squarefree; after dropping the divisibility and squarefreeness restrictions the sum is bounded by $\sum_{k=1}^{2} k^{1/2+\eta}(\sum_{r \geq 1} r^{-1-2\eta})^3$. For the remaining terms use (7.1) and $a = 3/2 + \eta$. Their contribution is

$$\ll_\eta t^{-3-2\eta} \sum_{m,n \geq 1} (mn)^{-1-\eta} \log(2mn) \ll_\eta t^{-3-2\eta} \leq T^2 t^{-1-2\eta}.$$

This proves the first assertion. Tonelli's theorem and Cauchy–Schwarz then give

$$\sum_{v \leq X} J_{j,c(\cdot)J(-v|\cdot)}(T) \ll_\eta T^2 \int_{1/T}^{\infty} t^{-1-\eta}\, dt \ll_\eta T^{2+\eta},$$

which is the second assertion. The argument used no reality condition on $c(n)$. □

The zero detector is also uniform in the character. Put $G_j(s) = \frac{1}{2}\pi^{-(s+j)/2}\Gamma((s+j)/2)$. If χ is a primitive nonprincipal character of conductor q and $\chi(-1) = (-1)^j$, the theta functional equation and Mellin formula give

$$\sqrt{q}\, t\, |\theta_{j,\chi}(t)| = |\theta_{j,\chi}(1/(qt))|, \qquad \int_0^{\infty} \theta_{j,\chi}(t) t^{s-1}\, dt = G_j(s) L(s,\chi). \tag{7.4}$$

The second identity holds for $\Re s > 0$ (indeed by continuation wherever both sides are defined). The first follows from Poisson summation; the conjugate character appearing in that formula has the same absolute value of its theta series, since g_j is real. If $q \leq T^2$, changing variables $u = 1/(qt)$ in the integral over $(0, T^{-1})$ therefore shows that

$$\int_0^{\infty} |\theta_{j,\chi}(t)|\, \frac{dt}{\sqrt{t}} \leq 2 J_{j,\chi}(T). \tag{7.5}$$

Lemma 7.2 (A zero forces a large tail)**.** *Let $\eta > 0$, $T \geq 1$, and let χ be as above, with $q \leq T^2$. If $J_{j,\chi}(T) + J_{j,\mu\chi}(T) \leq T^{2\eta}$, then every zero $s = \sigma + i\tau$ of $L(s,\chi)$ with $1/2 + 9\eta \leq \sigma \leq 1$ satisfies $|G_j(s)|^2 \ll T^{-2\eta}$, with an absolute implied constant.*

Proof. Let $M(s) = \int_{1/T}^{\infty} \theta_{j,\mu\chi}(t) t^{s-1}\, dt$ and consider $F(s) = G_j(s)^2 - G_j(s)L(s,\chi)M(s)$. All three Mellin transforms involved are holomorphic and bounded on $1/2 \leq \Re s \leq 2$: at infinity this follows from exponential decay, and at zero from the theta transformation for χ; M is cut off away from zero. On $\Re s = 1/2$, equations (7.4)–(7.5) imply $|F(s)| \ll 1 + J_{j,\chi}(T) J_{j,\mu\chi}(T) \ll T^{4\eta}$. On $\Re s = 2$, the absolutely convergent identity $\sum_n \mu(n)\chi(n)n^{-s} = 1/L(s,\chi)$ gives

$$F(s) = G_j(s) L(s,\chi) \int_0^{1/T} \theta_{j,\mu\chi}(t) t^{s-1}\, dt \ll T^{-1} \leq T^{4\eta-1}.$$

Here the uniform bounds $|G_j(s)L(s,\chi)| \ll 1$ and $|\theta_{j,\mu\chi}(t)| \ll 1/t$ suffice. Hadamard's three-lines theorem, with $\lambda = (\sigma - 1/2)/(3/2)$, now yields $|F(s)| \ll T^{4\eta-\lambda} \leq T^{-2\eta}$ when $\sigma \geq 1/2 + 9\eta$. At a zero of $L(s,\chi)$ one has $F(s) = G_j(s)^2$, proving the assertion. □

Proposition 7.3 (A fixed complex character)**.** *Fix a Dirichlet character ξ modulo N, a real $Y \geq 0$, and $0 < \eta < 1/18$. The number of primes $d \leq X$ with $d \equiv 7 \pmod 8$ and $d \nmid N$ for which $L(s, \xi\chi_d)$ has a zero in $\Re s \geq 1/2 + 9\eta$, $|\Im s| \leq Y$ is $O_{N,Y,\eta}(X^{1-\eta/2})$, where $\chi_d(n) = (n/d)$.*

Proof. Let ψ be the primitive character inducing ξ, of conductor $M \mid N$. Since $d \nmid N$, the product $\psi\chi_d$ is primitive of conductor Md and is nonprincipal. The Euler factors relating it to $\xi\chi_d$ are nonzero for $\Re s > 0$, so their zeros in the stated region coincide. Since $d \equiv 7 \pmod 8$, the parity j of $\psi\chi_d$ is fixed by ψ and quadratic reciprocity gives $\chi_d(n) = J(-d \mid n)$ for every $n \geq 1$; at 2 this uses $(2/d) = 1$.

Choose $T = \sqrt{MX}$. On the compact set $\{1/2 \leq \Re s \leq 1,\ |\Im s| \leq Y\}$, the nonvanishing gamma factor G_j has a positive minimum in absolute value. For all sufficiently large X, Lemma 7.2 and the Euler product in $\Re s > 1$ show that each exceptional prime d satisfies $J_{j,\psi\chi_d}(T) + J_{j,\mu\psi\chi_d}(T) > T^{2\eta}$. Both ψ and $\mu\psi$ have absolute value at most one, and $X \leq T^2$. Applying (7.3) to these two sequences and including all integers $v \leq X$ in the upper bound gives

$$\#\{\text{exceptional } d \leq X\}\, T^{2\eta} \ll_\eta T^{2+\eta}, \qquad \#\{\text{exceptional } d \leq X\} \ll_{N,Y,\eta} X^{1-\eta/2}.$$

The finitely many smaller values of X are absorbed in the implied constant. □

Corollary 7.4 (Simultaneous choice of a twist)**.** *Fix $\beta > 1/2$, $Y \geq 0$, and a finite set $\mathcal{X}$ of Dirichlet characters modulo N. Let Q be a multiple of $\operatorname{lcm}(8, N)$ and let a be a reduced residue class modulo Q with $a \equiv 7 \pmod 8$. There are infinitely many primes $d \equiv a \pmod Q$ such that every $L(s, \xi\chi_d)$, $\xi \in \mathcal{X}$, is nonzero for $\Re s \geq \beta$, $|\Im s| \leq Y$.*

Proof. Choose $0 < \eta < 1/18$ with $1/2 + 9\eta < \beta$. By Proposition 7.3, the union of the exceptional sets for $\xi \in \mathcal{X}$ has $O(X^{1-\eta/2})$ primes up to X. The prime number theorem in the fixed reduced residue class $a \pmod Q$ gives $(1 + o(1))X/(\varphi(Q) \log X)$ primes in that class. The latter count dominates the former as $X \to \infty$. □

This argument establishes the fixed-height twist selection used in the reduction, including complex ξ. It does not assert a zero-free region whose height increases with d, and its constants may depend on the fixed modulus and height. These are exactly the quantifiers needed when the twist is chosen before the subsequent limiting argument.

8. A one-sided criterion for zeros

We complete the proof by separating two issues. A one-sided estimate controls poles of a Mellin transform when its real residues have the appropriate sign. A quadratic twist, chosen at a fixed height, supplies that sign while ensuring that the pole attached to a proposed zero cannot cancel. The order of these choices avoids any uniformity assertion in the discriminant.

The following form of Landau's argument is classical; see [3, Section 15.1]. We include the short proof to specify the residue sign exactly.

Lemma 8.1 (Landau's one-sided principle)**.** *Let $E : [0, \infty) \to \mathbb{R}$ be locally integrable and $E(u) = O(e^u)$. Suppose $\Phi(s) = \int_0^\infty E(u) e^{(1-s)u}\, du$, initially for $\operatorname{Re} s > 2$, extends meromorphically to $\operatorname{Re} s > 0$, has at most simple poles there, and is holomorphic on $\operatorname{Re} s \geq 1$. Let $0 < \beta < 1$. If every residue at a real point $s > \beta$ is nonnegative and*

$$E(u) \leq C e^{(\beta-1)u} \qquad (u \geq u_0) \tag{8.1}$$

for some $C > 0$ and $u_0 \geq 0$, then Φ is holomorphic on $\operatorname{Re} s > \beta$.

Proof. The nonnegative function $g(u) = \mathbf{1}_{u \geq u_0}(Ce^{\beta u} - e^u E(u))$ has a Laplace transform $P(s)$ with a finite upper bound for its abscissa of convergence. Direct integration gives, on a right half-plane,

$$P(s) = \frac{Ce^{-(s-\beta)u_0}}{s - \beta} - \Phi(s) + \int_0^{u_0} E(u) e^{(1-s)u}\, du. \tag{8.2}$$

The last term is entire. Let α be the abscissa of convergence of P, permitting $\alpha = -\infty$. For a nonnegative integrand, a finite α is a singularity on the real axis. To recall the argument, if

P were analytic near α, choose $\sigma > \alpha$ close enough that its Taylor series converges at some $\sigma - h < \alpha$. Differentiation under the integral at σ gives $(-1)^k P^{(k)}(\sigma) = \int_0^\infty u^k g(u) e^{-\sigma u}\, du$. The Taylor series at $\sigma - h$ then has nonnegative terms, so monotone convergence makes its sum equal to $\int_0^\infty g(u) e^{-(\sigma-h)u}\, du < \infty$, contrary to the definition of α.

Suppose $\alpha > \beta$. The right side of (8.2) is meromorphic near α and its residue is the negative of the residue of Φ there. If that residue of Φ is positive, $P(s)$ tends to $-\infty$ as real $s \downarrow \alpha$, contradicting $P(s) \geq 0$. If it is zero, the singularity is removable, contradicting the preceding paragraph. This includes $\alpha \geq 1$, where Φ is holomorphic. Therefore $\alpha \leq \beta$. The defining integral of P is holomorphic on $\operatorname{Re} s > \beta$, and (8.2) proves the same for Φ. □

Proposition 8.2 (`entireL_ne_zero_of_primeError_le`). *Let M be divisible by 16, let a be a reduced residue class modulo M with $a \equiv 7 \pmod 8$, and let $1/2 < \beta < 1$. Suppose that for every prime $d \equiv a \pmod M$,*

$$R_{d,M}(x) \leq x^{\beta-1} \qquad \text{for all sufficiently large } x, \tag{8.3}$$

where the threshold may depend on d. Then $\widetilde{L}(s,\xi)$ has no zero in $\operatorname{Re} s > \beta$ for any character ξ modulo M.

Proof. Classical nonvanishing on $\operatorname{Re} s \geq 1$ reduces the assertion to $\beta < \operatorname{Re}\rho < 1$. Suppose that $\widetilde{L}(\rho,\xi_0) = 0$ for one character. By Corollary 7.4, choose a prime $d \equiv a \pmod M$ such that $L(s,\xi\chi_d)$ is nonzero on $\operatorname{Re} s \geq \beta$, $|\operatorname{Im} s| \leq |\operatorname{Im}\rho|$, for every character ξ modulo M. All these twists are nonprincipal.

For a real $t \in (\beta, 1)$, formula (2.6) now has numerator $\sum_\xi m_\xi(t) \geq 0$ and positive denominator $\varphi(M)t(1-t)$. The real residues of $\Phi_{d,M}$ in $t > \beta$ are therefore nonnegative; there are no poles for $t \geq 1$. Also $R_{d,M}(e^u) = O(e^u)$, since $0 \leq \lambda_{d,M}(n) \leq 2\log n \leq 2n$. Lemma 2.1 verifies the other transform hypotheses, and (8.3) verifies (8.1). Lemma 8.1 makes $\Phi_{d,M}$ holomorphic in $\operatorname{Re} s > \beta$.

At ρ, however, all twisted multiplicities in (2.6) vanish and $\sum_\xi m_\xi(\rho) \geq m_{\xi_0}(\rho) > 0$. Its denominator is nonzero. Hence the residue is nonzero, a contradiction. The selected d is fixed before taking $x \to \infty$ in (8.3), so no common threshold for all twists is required. □

Proof of Theorem 1.1. Proposition 3.4 proves the prime-sum assertion. Let χ be a character modulo q and suppose that $L(\rho,\chi) = 0$ with $\operatorname{Re}\rho > 1/2$, with the principal pole excluded. Classical nonvanishing allows us to assume $\operatorname{Re}\rho < 1$. Put $M = 18480q$ and $a = M/2 - 1$. Since $4 \mid M$, one has $a^2 \equiv 1 \pmod M$, so a is a unit. Since $16 \mid M$, one has $a \equiv 7 \pmod 8$.

Choose $1/2 < \beta < \operatorname{Re}\rho$. For every prime $d \equiv a \pmod M$, Proposition 3.4, with any $0 < \eta < \beta - 1/2$, implies (8.3). Induce χ to modulus M. Its L-function differs from $L(s,\chi)$ only by Euler factors of the form $1 - \chi^*(p)p^{-s}$, with χ^* the primitive inducing character; these are nonzero in $\operatorname{Re} s > 0$. Thus the induced L-function still vanishes at ρ, contrary to Proposition 8.2.

Finally, the functional equation of a primitive Dirichlet L-function sends a zero in $0 < \operatorname{Re} s < 1/2$ to a zero of the conjugate character in $\operatorname{Re} s > 1/2$. The gamma factors have no zeros or poles in the open critical strip. Imprimitive Euler factors are nonzero there, so the same assertion holds for every character. The modulus-one case gives the Riemann hypothesis for ζ. □

Appendix A. Arithmetic inputs and the formal source

The mathematical result established in this paper is Theorem 1.1, with Hypothesis 3.1 explicit. Sections 4, 5, and 6 further reduce its three arithmetic clauses to precise local, moment, and height estimates. The formal repository provides proofs of those inputs. Incorporating their complete human proofs would turn the present reduction into an unconditional manuscript following that source; simply suppressing the hypotheses would not do so.

The following list identifies the remaining construction work by mathematical role. A classification as construction-specific means that the exact statement is specialized to this argument; it does not establish a claim of historical priority.

Part	Required arithmetic content
Compatible data	One set of fields, levels, weights, top and thick places, lattices, and twists satisfying all the stated scales and orbit counts. The existence of numerical scales alone does not construct these objects.
Short forms	A seesaw identity with one twist-independent scalar; integral coefficient control away from the twist conductor; the jet-size estimate; and a small determinant index for the submodule repairing lost divisibility at the exceptional nonsplit primes.
Spanning	Almost isometry after deleting the bad types and a normalized second moment with leading constant 1 and error $9b^{-4}$; the ramification kernel bound must apply to the same column space and level.
Long forms	Prime-dependent twists satisfying all the local congruence conditions with summable budgets; integral comparison forms; the exact Hecke-value ratio bound; chain determinant factorization; and auxiliary forms completing the almost spanning family.

An economical organization is to write full proofs of these estimates and cite classical tools inside those proofs. For example, a general theta inner product formula supplies a relation between complex quantities, while the short-form argument also needs integral denominators and control at all embeddings. A general congruence theorem is useful only after its local conditions and constants are checked for the selected twists. Likewise, general results on Poincaré series do not imply the sharp moment estimate in (5.9). These quantitative specializations are the material a complete research paper must retain.

Here is the correspondence for the main propositions. Paths in this table are relative to `grh-compact/src/`; names refer to the pinned snapshot cited in the introduction.

Mathematical statement	Principal source
Short forms, Proposition 4.1	`New/Endgame/ShortAndEnough.lean`
Enough forms, Proposition 5.3	`New/Endgame/ShortAndEnough.lean`
Integral forms are long, Theorem 6.4	`New/Endgame/LongForms.lean`
The selected family, Proposition 3.2	`New/Endgame/LongForms.lean`
The main inequality, Proposition 3.3	`New/Main.lean`
The prime-sum error and its transform	`New/Endgame/PrimeError.lean`
One-sided zero-free criterion	`New/Endgame/OneSidedCriterion.lean`
Quadratic-twist selection	`Lib/Analytic/QuadraticTwist/ZeroFreeTwist.lean`
GRH in half-plane form	`New/Main.lean`: `New.grh`

For the inputs to short forms, the main modules are `New/Columns/LocalStatement.lean` and `New/Seesaw/SeesawTheorem.lean`. For spanning they are `New/Filling/Moments.lean` and the local kernel and energy estimates it imports. For the lower bound they include `New/Selmer/FitTwistsExist.lean`, `New/Rows/LiftVolumeHeight.lean`, `New/Rows/RowVolume.lean`, `New/Heights/OneDeterminant.lean`, and `New/Exchange/CrudeRows.lean`. The compatible-data existence statement is in `New/Data/DataExist.lean`. The fixed-modulus prime number theorem route used in Proposition 3.4 follows the readable edition's `grh/src/New/Reduction/OneSidedBound.lean`; the compact edition uses the localized window argument described after that proposition.

The elementary proofs above agree with the numerical constants and quantifiers of the inspected source. This does not constitute a fresh compilation or a full audit of its transitive dependencies. The repository reports successful machine checks with Lean 4.35.0 and Mathlib commit `56a005855ea17e0b0958f73ec801ae140b0a1679`; its stated status is a research artifact

with machine checks and no completed human review. Those are claims of the repository, not additional verification performed by this manuscript. Its `REFERENCES.md` also distinguishes citations checked against a precise statement from references for which only the work was located.

For publication, the remaining boundary is therefore exact: establish the compatible construction and the three arithmetic estimates, with the normalizations used here. The prime-sum passage, the finite-dimensional selection, the determinant algebra, and the fixed-height complex-twist argument have been supplied. None of these remaining arithmetic estimates has been labelled a known theorem merely because the source places a related result in a background directory.

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